

FORMALITY PROPERTIES OF FINITELY GENERATED GROUPS AND LIE ALGEBRAS

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ABSTRACT. We explore the graded and filtered formality properties of finitely generated groups by studying the various Lie algebras over a field of characteristic 0 attached to such groups, including the Malcev Lie algebra, the associated graded Lie algebra, the holonomy Lie algebra, and the Chen Lie algebra. We explain how these notions behave with respect to split injections, coproducts, direct products, as well as field extensions, and how they are inherited by solvable and nilpotent quotients. A key tool in this analysis is the 1-minimal model of the group, and the way this model relates to the aforementioned Lie algebras. Another approach to formality is provided by Taylor expansions from the group to the completion of the associated graded algebra of the group ring. We illustrate our approach with examples drawn from a variety of group-theoretic and topological contexts, such as finitely generated torsion-free nilpotent groups, link groups, and fundamental groups of Seifert fibered manifolds.

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1. INTRODUCTION

The main focus of this paper is on the formality properties of finitely generated groups, as reflected in the structure of the various graded or filtered Lie algebras, as well as commutative, differential graded algebras attached to such groups.

1.1. From groups to Lie algebras. Throughout, we will let G be a finitely generated group, and we will let \mathbb{k} be a coefficient field of characteristic 0. Our main focus will be on several \mathbb{k} -Lie algebras attached to such a group, and the way they all connect to each other.

By far the best known of these Lie algebras is the *associated graded Lie algebra*, $\mathrm{gr}(G; \mathbb{k})$, introduced by P. Hall, W. Magnus, and E. Witt in the 1930s, cf. [54]. This is a finitely generated graded Lie algebra, whose graded pieces are the successive quotients of the lower central series of G (tensored with \mathbb{k}), and whose Lie bracket is induced from the group commutator. The quintessential example is the free Lie algebra $\mathrm{lie}(\mathbb{k}^n)$, which is the associated graded Lie algebra of the free group on n generators, F_n .

Closely related is the *holonomy Lie algebra*, $\mathfrak{h}(G; \mathbb{k})$, introduced by T. Kohno in [41], building on work of K.-T. Chen [15], and further studied by Markl–Papadima [56] and Papadima–Suciu [63]. This is a quadratic Lie algebra, obtained as the quotient of the free Lie algebra on $H_1(G; \mathbb{k})$ by the ideal generated by the image of the dual of the cup product map in degree 1. The holonomy Lie algebra comes equipped with a natural epimorphism $\Phi_G: \mathfrak{h}(G; \mathbb{k}) \twoheadrightarrow \mathrm{gr}(G; \mathbb{k})$, and thus can be viewed as the quadratic approximation to the associated graded Lie algebra.

The most intricate of these Lie algebras (yet, in many ways, the most important) is the *Malcev Lie algebra*, $\mathfrak{m}(G; \mathbb{k})$. As shown by A. Malcev in [55], every finitely generated, torsion-free nilpotent group N is the fundamental group of a nilmanifold, whose corresponding \mathbb{k} -Lie algebra is $\mathfrak{m}(N; \mathbb{k})$. Taking now the nilpotent quotients of G , we may define $\mathfrak{m}(G; \mathbb{k})$ as the inverse limit of the resulting tower of nilpotent Lie algebras, $\mathfrak{m}(G/\Gamma_k G; \mathbb{k})$. By construction, the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra. The pronilpotent group corresponding to this pronilpotent Lie algebra is denoted by $\mathfrak{M}(G; \mathbb{k})$.

In two seminal papers, [74, 73], D. Quillen showed that the Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is the set of all primitive elements in $\widehat{\mathbb{k}G}$ (the completion of the group algebra of G with respect to the filtration by powers of the augmentation ideal), and that the associated graded Lie algebra of $\mathfrak{m}(G; \mathbb{k})$ with respect to the inverse limit filtration is isomorphic to $\mathrm{gr}(G; \mathbb{k})$. Furthermore, the set of all group-like elements in $\widehat{\mathbb{k}G}$, with multiplication and filtration inherited from $\widehat{\mathbb{k}G}$, forms a complete, filtered group isomorphic to $\mathfrak{M}(G; \mathbb{k})$.

1.2. Formality notions. In his foundational paper on rational homotopy theory [83], D. Sullivan associated to each path-connected space X a ‘minimal model’, $\mathcal{M}(X)$, which can be viewed as an algebraic approximation to the space. If, moreover, X is a CW-complex with finitely many 1-cells, then the Lie algebra dual to the first stage of the minimal model is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ associated to the fundamental group $G = \pi_1(X)$.

The space X is said to be *formal* if the commutative, graded differential algebra $\mathcal{M}(X)$ is quasi-isomorphic to the cohomology ring $H^*(X; \mathbb{Q})$, endowed with the zero differential. If there exists a DGA morphism from the i -minimal model $\mathcal{M}(X, i)$ to $H^*(X; \mathbb{Q})$ inducing isomorphisms in cohomology up to degree i and a monomorphism in degree $i + 1$, then X is called *i -formal*.

A finitely generated group G is said to be *1-formal* (over \mathbb{Q}) if it has a classifying space $K(G, 1)$ which is 1-formal. The study of the various Lie algebras attached to the fundamental group of a

space provides a fruitful way to look at the formality problem. Indeed, the group G is 1-formal if and only if the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{Q})$ is isomorphic to the rational holonomy Lie algebra of G , completed with respect to the lower central series (LCS) filtration.

We find it useful to separate the 1-formality property of a group G into two complementary properties: graded formality and filtered formality. More precisely, we say that G is *graded-formal* (over \mathbb{k}) if the associated graded Lie algebra $\mathrm{gr}(G; \mathbb{k})$ is isomorphic, as a graded Lie algebra, to the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$. Likewise, we say that G is *filtered-formal* (over \mathbb{k}) if the Malcev Lie algebra $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ is isomorphic, as a filtered Lie algebra, to the completion of its associated graded Lie algebra, $\widehat{\mathrm{gr}}(\mathfrak{m})$, where both \mathfrak{m} and $\widehat{\mathrm{gr}}(\mathfrak{m})$ are endowed with the respective inverse limit filtrations. As we show in Proposition 7.6, the group G is 1-formal if and only if it is both graded-formal and filtered-formal.

Based on the work in papers of Berceanu–Papadima [5], Dimca et al. [19], Lambe–Priddy [44], Hain [32], Massey [57], Matei–Suciu [59], we summarize the following examples to show that all four possible combinations of these formality properties occur:

- (1) Examples of 1-formal groups include finitely generated free groups and free abelian groups (more generally, right-angled Artin groups), groups with first Betti number equal to 0 or 1, fundamental groups of compact Kähler manifolds, and fundamental groups of complements of complex algebraic hypersurfaces.
- (2) There are many torsion-free, nilpotent groups (Examples 7.8 and 10.8) as well as fundamental groups of link complements (Examples 7.11 and 7.12) which are filtered-formal, but not graded-formal.
- (3) There are also finitely presented groups, such as those from Examples 7.9, 7.10, and 7.13 which are graded-formal but not filtered-formal.
- (4) Finally, there are groups which enjoy none of these formality properties. Indeed, if G_1 is one of the groups from (2) and G_2 is one of the groups from (3), then Theorem 1.5 below shows that the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.

1.3. Field extensions and formality. We start by reviewing in §2 some basic notions pertaining to filtered and graded Lie algebras. We say that a Lie algebra \mathfrak{g} (over a field \mathbb{k} of characteristic 0) is *filtered formal* if it admits complete, separated filtration, and there exists a filtration-preserving isomorphism $\mathfrak{g} \cong \widehat{\mathrm{gr}}(\mathfrak{g})$ which induces the identity on associated graded Lie algebras. Our first result, which generalizes a recent theorem of Cornulier [16], shows that filtered-formality behaves well with respect to field extensions. The proof we give in Theorem 2.8 is based on recent work of Enriquez [23] and Maassarani [52].

Theorem 1.1. *Let \mathfrak{g} be a \mathbb{k} -Lie algebra endowed with a complete, separated filtration such that $\mathrm{gr}(\mathfrak{g})$ is finitely generated in degree 1. If $\mathbb{k} \subset \mathbb{K}$ is a field extension, then \mathfrak{g} is filtered-formal if and only if the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal.*

We continue in §3 with a review of the notions of quadratic and Koszul algebras. In §4, we analyze in detail the relationship between the 1-minimal model $\mathcal{M}(A, 1)$ and the dual Lie algebra $\mathfrak{Q}(A)$ of a differential graded \mathbb{k} -algebra (A, d) . The reason for doing this is a result of Sullivan [83], which gives a functorial isomorphism of pronilpotent Lie algebras, $\mathfrak{Q}(A) \cong \mathfrak{m}(G; \mathbb{k})$, provided $\mathcal{M}(A, 1)$ is a 1-minimal model for a finitely generated group G .

Of particular interest is the case when A is a connected, graded commutative algebra with finite dimensional A^1 , endowed with the zero differential. In Theorem 4.10, we show that $\mathfrak{L}(A)$ is isomorphic (as a filtered Lie algebra) to the degree completion of the holonomy Lie algebra of A . In the case when $A^{\leq 2} = H^{\leq 2}(G; \mathbb{k})$, for some finitely generated group G , this result recovers the aforementioned characterization of the 1-formality property of G . In Theorem 6.5 we give an alternate interpretation of filtered formality, which will be used in the proof of Corollaries 11.3 and 11.4.

Theorem 1.2. *A group G is filtered-formal if and only if G has a 1-minimal model whose differential is homogeneous with respect to the canonical Hirsch weights.*

As is well-known, a space X with finite Betti numbers is formal over \mathbb{Q} if and only if it is formal over \mathbb{k} , for any field \mathbb{k} of characteristic 0. This foundational result was proved independently and in various degrees of generality by Halperin and Stasheff [35], Neisendorfer and Miller [61], and Sullivan [83]. Motivated by these classical results, as well as the aforementioned work of Cornulier, we investigate the way in which the formality properties of spaces and groups behave under field extensions. Our next result, which is a combination of Corollary 4.23, Corollary 5.10, Proposition 6.6, and Corollary 7.7, can be stated as follows.

Theorem 1.3. *Let X be a path-connected space, with finitely generated fundamental group G . Let \mathbb{k} be a field of characteristic zero, and let $\mathbb{K} \subset \mathbb{K}$ be a field extension.*

- (1) *Suppose X has finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i -formal over \mathbb{k} if and only if X is i -formal over \mathbb{K} .*
- (2) *The group G is 1-formal over \mathbb{k} if and only if G is 1-formal over \mathbb{K} .*
- (3) *The group G is graded-formal over \mathbb{k} if and only if G is graded-formal over \mathbb{K} .*
- (4) *The group G is filtered-formal over \mathbb{k} if and only if G is filtered-formal over \mathbb{K} .*

In summary, under appropriate finiteness conditions, all the formality properties that we study in this paper are independent of the ground field $\mathbb{k} \supset \mathbb{Q}$. Hence, we will sometimes avoid mentioning the coefficient field when referring to these formality notions. The descent property for partial formality from Theorem 1.3, part (1) has been used in [66] to establish the $(n-1)$ -formality over \mathbb{Q} of compact Sasakian manifolds of dimension $2n+1$.

1.4. Propagation of formality. Next, we turn our attention to the way in which the various formality notions for groups behave with respect to split injections, coproducts, and direct products. Our first result in this direction is a combination of Theorem 5.11 and 7.16, and can be stated as follows.

Theorem 1.4. *Let G be a finitely generated group, and let $K \leq G$ be a subgroup. Suppose there is a split monomorphism $\iota: K \rightarrow G$. Then:*

- (1) *If G is graded-formal, then K is also graded-formal.*
- (2) *If G is filtered-formal, then K is also filtered-formal.*
- (3) *If G is 1-formal, then K is also 1-formal.*

In particular, if a semi-direct product $G_1 \rtimes G_2$ has one of the above formality properties, then G_2 also has that property; in general, though, G_1 will not, as illustrated in Example 5.13.

As shown by Dimca et al. [19], both the product and the coproduct of two 1-formal groups is again 1-formal. Also, as shown by Plantiko [69], the product and coproduct of two graded-formal groups is again graded-formal. We sharpen these results in the next theorem, which is a combination of Propositions 5.15 and 7.18.

Theorem 1.5. *Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.*

- (1) G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).
- (2) $G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).
- (3) $G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).

Both Theorem 1.4 and Theorem 1.5 can be used to decide the formality properties of new groups from those of known groups. In general, though, even when both G_1 and G_2 are 1-formal, we cannot conclude that an arbitrary semi-direct product $G_1 \rtimes G_2$ is 1-formal (see Example 7.13).

The various formality properties are not necessarily inherited by quotient groups. However, as we shall see in Theorem 1.7 and Theorem 7.14, respectively, filtered formality is passed on to the derived quotients and to the nilpotent quotients of a group.

1.5. Expansions of groups and formality. In §8, we relate the 1-formality and filtered formality properties of a group to the Taylor expansions of its group algebra. Expansions of free groups were first introduced by Magnus, cf. [54]. This technique has been generalized and used in many ways, for instance, to give a presentation for the Malcev Lie algebra of a finitely presented group, [58, 62]. X.-S. Lin studied in [50] expansions of fundamental groups of smooth manifolds, using K.T. Chen's theory of formal power series connections and their induced monodromy representations. More generally, D. Bar-Natan has explored in [3] the Taylor expansion of an arbitrary ring.

For a finitely generated group G , let $\text{gr}(\mathbb{k}G)$ be the associated graded algebra of $\mathbb{k}G$ with respect to the augmentation ideal, and let $\widehat{\text{gr}}(\mathbb{k}G)$ be the degree completion of this algebra. Developing an idea from [3], we say that a map $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is a multiplicative expansion of G if the induced algebra morphism, $\bar{E}: \mathbb{k}G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, is filtration-preserving and induces the identity on associated graded algebras. Such a map E is called a *Taylor expansion* if it sends all elements of G to group-like elements of the Hopf algebra $\widehat{\text{gr}}(\mathbb{k}G)$.

A group G is said to be *residually torsion-free nilpotent* if any non-trivial element of G can be detected in a torsion-free nilpotent quotient. If G is finitely generated, this condition is equivalent to the injectivity of the canonical map to the Malcev group completion, $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$. The following result, which is a combination of Theorem 8.5 and Corollaries 8.6 and 8.8.

Theorem 1.6. *Let G be a finitely generated group G . Then:*

- (1) G is filtered-formal if and only if G has a Taylor expansion $G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$.
- (2) G is 1-formal if and only if G has a Taylor expansion and $\text{gr}(\mathbb{k}G)$ is a quadratic algebra.
- (3) G has an injective Taylor expansion if and only if G is residually torsion-free nilpotent and filtered-formal.

Combining this theorem with our other results on filtered formality, we conclude that the existence of a Taylor expansion is preserved under field extensions, finite products and coproducts, split injections, nilpotent quotients, and solvable quotients of groups. In particular, if a finitely generated group G has a Taylor expansion over \mathbb{C} , then it also has a Taylor expansion over \mathbb{Q} .

1.6. Derived series and Lie algebras. In §9, we investigate some of the relationships between the lower central series and derived series of a group, on one hand, and the derived series of the corresponding Lie algebras, on the other hand.

In [14], Chen studied the lower central series quotients of the maximal metabelian quotient of a finitely generated free group, and computed their graded ranks. More generally, following Papadima

and Suciu [63], we may define the *Chen Lie algebras* of a group G as the associated graded Lie algebras of its solvable quotients, $\mathrm{gr}(G/G^{(i)}; \mathbb{k})$. Our next theorem (which combines Theorem 9.3 and Corollary 9.5) sharpens and extends the main result of [63].

Theorem 1.7. *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$\Psi_G^{(i)} : \mathrm{gr}(G; \mathbb{k}) / \mathrm{gr}(G; \mathbb{k})^{(i)} \twoheadrightarrow \mathrm{gr}(G/G^{(i)}; \mathbb{k}) .$$

Moreover,

- (1) *If G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.*
- (2) *If G is a 1-formal group, then $\mathfrak{h}(G; \mathbb{k}) / \mathfrak{h}(G; \mathbb{k})^{(i)} \cong \mathrm{gr}(G/G^{(i)}; \mathbb{k})$.*

Given a finitely presented group G , the solvable quotients $G/G^{(i)}$ need not be finitely presented. Thus, finding presentations for the Chen Lie algebra $\mathrm{gr}(G/G^{(i)})$ can be an arduous task. Nevertheless, Theorem 1.7 provides a method for finding such presentations, under suitable formality assumptions. The theorem can also be used as an obstruction to 1-formality.

1.7. Nilpotent groups and Lie algebras. Our techniques apply especially well to the class of finitely generated, torsion-free nilpotent groups. Carlson and Toledo [11] studied the 1-formality properties of such groups, while Plantiko [69] gave a sufficient conditions for such groups to be non-graded-formal. For nilpotent Lie algebras, the notion of filtered-formality has been studied by Leger [48], Cornulier [16], Kasuya [40], and others. In particular, Cornulier [16] proves that the systolic growth of a finitely generated nilpotent group G is asymptotically equivalent to its growth if and only if the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is filtered-formal (or, ‘Carnot’), while Kasuya [40] shows that the variety of flat connections on a filtered-formal (or, ‘naturally graded’), n -step nilpotent Lie algebra has a singularity at the origin cut out by polynomials of degree at most $n + 1$.

We investigate in §10 the filtered formality of nilpotent groups and Lie algebras. The next result combines Theorem 10.4 and Proposition 10.9.

Theorem 1.8. *Let G be a finitely generated, torsion-free nilpotent group.*

- (1) *Suppose G is a 2-step nilpotent group with torsion-free abelianization. Then G is filtered-formal.*
- (2) *Suppose G is filtered-formal. Then the universal enveloping algebra $U(\mathrm{gr}(G; \mathbb{k}))$ is Koszul if and only if G is abelian.*

As mentioned previously, nilpotent quotients of finitely generated filtered-formal groups are filtered-formal. In particular, each n -step, free nilpotent group $F/\Gamma_n F$ is filtered-formal. A classical example is the unipotent group $U_n(\mathbb{Z})$, which is known to be filtered-formal by Lambe and Priddy [44], but not graded-formal for $n \geq 3$.

In [16], Cornulier showed that the filtered-formality of a finite-dimensional nilpotent Lie algebra is independent of the ground field, thereby answering a question of Johnson [39]. Theorem 1.1, which holds in a much more general context, allows us to recover this result, see Corollary 10.3.

1.8. Further applications. We end in §11 with a detailed study of fundamental groups of (orientable) Seifert fibered manifolds from a rational homotopy viewpoint. Starting from the minimal model of such a manifold M , as described in [72], we find a presentation for the Malcev Lie algebra

$m(\pi_1(M); \mathbb{k})$, and we use this information to derive a presentation for $\text{gr}(\pi_1(M); \mathbb{k})$. As an application, we show that Seifert manifold groups are filtered-formal, and determine precisely which ones are graded-formal. The techniques developed here have been used in [66] to prove a more general result about the filtered formality of Sasakian groups.

This work was motivated in good part by the papers [2, 10] of Etingof et al. on the triangular and quasi-triangular groups, also known as the (upper) pure virtual braid groups. In [79], we apply the techniques developed in this paper to study the formality properties of such groups. Related results for the McCool groups (also known as the welded pure braid groups) and other braid-like groups are given in [81, 78].

2. FILTERED AND GRADED LIE ALGEBRAS

In this section we study the interactions between filtered Lie algebras, their completions, and their associated graded Lie algebras, mainly as they relate to the notion of filtered formality.

2.1. Graded Lie algebras. We start by reviewing some standard material on Lie algebras, following the exposition from the works of Ekedahl and Merkulov [22], Polishchuk and Positselski [70], Quillen [73], and Serre [76].

Fix a ground field \mathbb{k} of characteristic 0. Let \mathfrak{g} be a Lie algebra over \mathbb{k} , i.e., a \mathbb{k} -vector space \mathfrak{g} endowed with a bilinear operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Lie identities. We say that \mathfrak{g} is a *graded Lie algebra* if \mathfrak{g} decomposes as $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$ and the Lie bracket sends $\mathfrak{g}_i \times \mathfrak{g}_j$ to \mathfrak{g}_{i+j} , for all i and j . A morphism of graded Lie algebras is a \mathbb{k} -linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ which preserves the Lie brackets and the degrees. In particular, φ induces \mathbb{k} -linear maps $\varphi_i: \mathfrak{g}_i \rightarrow \mathfrak{h}_i$ for all $i \geq 1$.

The most basic example of a graded Lie algebra is constructed as follows. Let V a \mathbb{k} -vector space. The tensor algebra $T(V)$ has a natural Hopf algebra structure, with comultiplication Δ and counit ε the algebra maps given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\varepsilon(v) = 0$, for $v \in V$. The *free Lie algebra* on V is the set of primitive elements, i.e.,

$$(1) \quad \text{lie}(V) = \{x \in T(V) \mid \Delta(x) = x \otimes 1 + 1 \otimes x\},$$

with Lie bracket $[x, y] = x \otimes y - y \otimes x$ and grading induced from $T(V)$.

A Lie algebra \mathfrak{g} is said to be *finitely generated* if there is an epimorphism $\varphi: \text{lie}(V) \rightarrow \mathfrak{g}$ for some finite-dimensional \mathbb{k} -vector space V . If, moreover, the Lie ideal $\mathfrak{r} = \ker \varphi$ is finitely generated as a Lie algebra, then \mathfrak{g} is called *finitely presented*.

Now suppose all elements of V are assigned degree 1 in $T(V)$. Then the inclusion $\iota: \text{lie}(V) \rightarrow T(V)$ identifies $\text{lie}_1(V)$ with $T_1(V) = V$. Furthermore, ι maps $\text{lie}_2(V)$ to $T_2(V) = V \otimes V$ by sending $[v, w]$ to $v \otimes w - w \otimes v$ for each $v, w \in V$; we thus may identify $\text{lie}_2(V) \cong V \wedge V$ by sending $[v, w]$ to $v \wedge w$.

If $\mathfrak{g} = \text{lie}(V)/\mathfrak{r}$, with V a (finite-dimensional) vector space concentrated in degree 1, then we say \mathfrak{g} is *(finitely) generated in degree 1*. If, moreover, the Lie ideal \mathfrak{r} is homogeneous, then \mathfrak{g} is a graded Lie algebra. In particular, if \mathfrak{g} is finitely generated in degree 1 and the homogeneous ideal \mathfrak{r} is generated in degree 2, then we say \mathfrak{g} is a *quadratic Lie algebra*. The next lemma is standard; we include a quick proof for completeness.

Lemma 2.1. *Let $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}$ be an endomorphism a graded Lie algebra. Suppose \mathfrak{g} is finitely generated in degree 1, and the restriction $\varphi_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ is an isomorphism. Then φ is an isomorphism.*

Proof. Since \mathfrak{g} is generated in degree 1, and since φ_1 is an isomorphism, each linear map $\varphi_n: \mathfrak{g}_n \rightarrow \mathfrak{g}_n$ is surjective. Furthermore, since \mathfrak{g} is finitely generated, we have that $\dim(\mathfrak{g}_n) < \infty$, for all $n \geq 1$. Hence, each map φ_n is an isomorphism, and thus φ itself is an isomorphism. \square

2.2. Filtrations. We will be very much interested in this work in Lie algebras endowed with a filtration, usually but not always enjoying an extra ‘multiplicative’ property. At the most basic level, a *filtration* \mathcal{F} on a Lie algebra \mathfrak{g} is a nested sequence of Lie ideals, $\mathfrak{g} = \mathcal{F}_1\mathfrak{g} \supset \mathcal{F}_2\mathfrak{g} \supset \cdots$.

A well-known such filtration is the *derived series*, $\mathcal{F}_i\mathfrak{g} = \mathfrak{g}^{(i-1)}$, defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \geq 1$. The derived series is preserved by Lie algebra maps. The quotient Lie algebras $\mathfrak{g}/\mathfrak{g}^{(i)}$ are solvable; moreover, if \mathfrak{g} is a graded Lie algebra, all these solvable quotients inherit a graded Lie algebra structure.

The existence of a filtration \mathcal{F} on a Lie algebra \mathfrak{g} makes \mathfrak{g} into a topological vector space, by defining a basis of open neighborhoods of an element $x \in \mathfrak{g}$ to be $\{x + \mathcal{F}_k\mathfrak{g}\}_{k \in \mathbb{N}}$. The fact that each basis neighborhood $\mathcal{F}_k\mathfrak{g}$ is a Lie subalgebra implies that the Lie bracket map $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is continuous; thus, \mathfrak{g} is, in fact, a topological Lie algebra. We say that \mathfrak{g} is *complete* (respectively, *separated*) if the underlying topological vector space enjoys those properties.

Given an ideal $\mathfrak{a} \subset \mathfrak{g}$, there is an induced filtration on it, given by $\mathcal{F}_k\mathfrak{a} = \mathcal{F}_k\mathfrak{g} \cap \mathfrak{a}$. Likewise, the quotient Lie algebra, $\mathfrak{g}/\mathfrak{a}$, has a naturally induced filtration with terms $\mathcal{F}_k\mathfrak{g}/\mathcal{F}_k\mathfrak{a}$. Let $\bar{\mathfrak{a}}$ be the closure of \mathfrak{a} in the filtration topology. Then $\bar{\mathfrak{a}}$ is a closed ideal of \mathfrak{g} . Moreover, by the continuity of the Lie bracket, we have that

$$(2) \quad \overline{[\bar{\mathfrak{a}}, \bar{\mathfrak{r}}]} = \overline{[\mathfrak{a}, \mathfrak{r}]}.$$

Finally, if \mathfrak{g} is complete (or separated), then $\mathfrak{g}/\bar{\mathfrak{a}}$ is also complete (or separated).

2.3. Completions. For each $j \geq k$, there is a canonical projection $\mathfrak{g}/\mathcal{F}_j\mathfrak{g} \rightarrow \mathfrak{g}/\mathcal{F}_k\mathfrak{g}$, compatible with the projections from \mathfrak{g} to its quotient Lie algebras $\mathfrak{g}/\mathcal{F}_k\mathfrak{g}$. The *completion* of the Lie algebra \mathfrak{g} with respect to the filtration \mathcal{F} is defined as the limit of this inverse system, i.e.,

$$(3) \quad \widehat{\mathfrak{g}} := \varprojlim_k \mathfrak{g}/\mathcal{F}_k\mathfrak{g} = \left\{ (g_1, g_2, \dots) \in \prod_{i=1}^{\infty} \mathfrak{g}/\mathcal{F}_i\mathfrak{g} \mid g_j \equiv g_k \pmod{\mathcal{F}_k\mathfrak{g}} \text{ for all } j > k \right\}.$$

Using the fact that $\mathcal{F}_k(\mathfrak{g})$ is an ideal of \mathfrak{g} , it is readily seen that $\widehat{\mathfrak{g}}$ is a Lie algebra, with Lie bracket defined componentwise. Furthermore, $\widehat{\mathfrak{g}}$ has a natural inverse limit filtration, $\widehat{\mathcal{F}}$, given by

$$(4) \quad \widehat{\mathcal{F}}_k\widehat{\mathfrak{g}} := \widehat{\mathcal{F}_k\mathfrak{g}} = \varprojlim_{i \geq k} \mathcal{F}_k\mathfrak{g}/\mathcal{F}_i\mathfrak{g} = \{ (g_1, g_2, \dots) \in \widehat{\mathfrak{g}} \mid g_i = 0 \text{ for all } i < k \}.$$

Note that $\widehat{\mathcal{F}}_k\widehat{\mathfrak{g}} = \overline{\mathcal{F}_k\mathfrak{g}}$, and so each term of the filtration $\widehat{\mathcal{F}}$ is a closed Lie ideal of $\widehat{\mathfrak{g}}$. Furthermore, the Lie algebra $\widehat{\mathfrak{g}}$, endowed with this filtration, is both complete and separated.

Let $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ be the canonical map to the completion. Then ι is a morphism of Lie algebras, preserving the respective filtrations. Clearly, $\ker(\iota) = \bigcap_{k \geq 1} \mathcal{F}_k\mathfrak{g}$. Hence, ι is injective if and only if \mathfrak{g} is separated. Furthermore, ι is surjective if and only if \mathfrak{g} is complete.

2.4. Filtered Lie algebras. A *filtered Lie algebra* (over the field \mathbb{k}) is a Lie algebra \mathfrak{g} endowed with a \mathbb{k} -vector filtration $\{\mathcal{F}_k\mathfrak{g}\}_{k \geq 1}$ satisfying the ‘multiplicativity’ condition

$$(5) \quad [\mathcal{F}_r\mathfrak{g}, \mathcal{F}_s\mathfrak{g}] \subseteq \mathcal{F}_{r+s}\mathfrak{g}$$

for all $r, s \geq 1$. Obviously, this condition implies that each subspace $\mathcal{F}_k \mathfrak{g}$ is a Lie ideal, and so, in particular, \mathcal{F} is a Lie algebra filtration. Let

$$(6) \quad \text{gr}^{\mathcal{F}}(\mathfrak{g}) := \bigoplus_{k \geq 1} \mathcal{F}_k \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g}.$$

be the associated graded vector space to the filtration \mathcal{F} on \mathfrak{g} . Condition (5) implies that the Lie bracket map on \mathfrak{g} descends to a map $[\cdot, \cdot]: \text{gr}^{\mathcal{F}}(\mathfrak{g}) \times \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{F}}(\mathfrak{g})$, which makes $\text{gr}^{\mathcal{F}}(\mathfrak{g})$ into a graded Lie algebra, with graded pieces given by decomposition (6).

If \mathfrak{g} is a filtered Lie algebra, then its completion, $\widehat{\mathfrak{g}}$, is again a filtered Lie algebra. Indeed, if \mathcal{F} is the given multiplicative filtration on \mathfrak{g} , and $\widehat{\mathcal{F}}$ is the completed filtration on $\widehat{\mathfrak{g}}$, then $\widehat{\mathcal{F}}$ also satisfies property (5).

A morphism of filtered Lie algebras is a linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ preserving Lie brackets and the given filtrations, \mathcal{F} and \mathcal{G} . Such a morphism induces morphisms between the respective nilpotent quotients, $\varphi_k: \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$, and a morphism of associated graded Lie algebras, $\text{gr}(\varphi): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{G}}(\mathfrak{h})$.

By construction, the canonical map to the completion, $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$, is a morphism of filtered Lie algebras. It is readily seen that the induced morphism, $\text{gr}(\iota): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\widehat{\mathcal{F}}}(\widehat{\mathfrak{g}})$, is an isomorphism. Moreover, if \mathfrak{g} is both complete and separated, then the map $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ itself is an isomorphism of filtered Lie algebras.

Lemma 2.2. *Let $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ be a morphism of complete, separated, filtered Lie algebras, and suppose $\text{gr}(\varphi): \text{gr}^{\mathcal{F}}(\mathfrak{g}) \rightarrow \text{gr}^{\mathcal{G}}(\mathfrak{h})$ is an isomorphism. Then φ is also an isomorphism.*

Proof. By assumption, the homomorphisms $\text{gr}_k(\varphi): \mathcal{F}_k \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathcal{G}_k \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$ are isomorphisms, for all $k \geq 1$. An easy induction on k shows that all maps $\varphi_k: \mathfrak{g} / \mathcal{F}_{k+1} \mathfrak{g} \rightarrow \mathfrak{h} / \mathcal{G}_{k+1} \mathfrak{h}$ are isomorphisms. Therefore, the map $\widehat{\varphi}: \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{h}}$ is an isomorphism. On the other hand, both \mathfrak{g} and \mathfrak{h} are complete and separated, and so $\mathfrak{g} = \widehat{\mathfrak{g}}$ and $\mathfrak{h} = \widehat{\mathfrak{h}}$. Hence $\varphi = \widehat{\varphi}$, and we are done. \square

2.5. The degree completion. Any Lie algebra \mathfrak{g} comes equipped with a lower central series (LCS) filtration, $\{\Gamma_k(\mathfrak{g})\}_{k \geq 1}$, defined by $\Gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\Gamma_k(\mathfrak{g}) = [\Gamma_{k-1}(\mathfrak{g}), \mathfrak{g}]$ for $k \geq 2$. Clearly, this is a multiplicative filtration. Any other such filtration $\{\mathcal{F}_k(\mathfrak{g})\}_{k \geq 1}$ on \mathfrak{g} is coarser than this filtration; that is, $\Gamma_k \mathfrak{g} \subseteq \mathcal{F}_k \mathfrak{g}$, for all $k \geq 1$. Any Lie algebra morphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ preserves LCS filtrations. Furthermore, the quotient Lie algebras $\mathfrak{g} / \Gamma_k \mathfrak{g}$ are nilpotent. For simplicity, we shall write $\text{gr}(\mathfrak{g}) := \text{gr}^{\Gamma}(\mathfrak{g})$ for the associated graded Lie algebra and $\widehat{\mathfrak{g}}$ for the completion of \mathfrak{g} with respect to the LCS filtration Γ . Furthermore, we shall take $\widehat{\Gamma}_k = \overline{\Gamma}_k$ as the canonical filtration on $\widehat{\mathfrak{g}}$.

Every graded Lie algebra, $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$, has a canonical decreasing filtration induced by the grading, $\mathcal{F}_k \mathfrak{g} = \bigoplus_{i \geq k} \mathfrak{g}_i$. Moreover, if \mathfrak{g} is generated in degree 1, then this filtration coincides with the LCS filtration $\Gamma_k(\mathfrak{g})$. In particular, the associated graded Lie algebra with respect to \mathcal{F} coincides with \mathfrak{g} . In this case, the completion of \mathfrak{g} with respect to the lower central series (or, degree) filtration is called the *degree completion* of \mathfrak{g} , and is simply denoted by $\widehat{\mathfrak{g}}$. It is readily seen that $\widehat{\mathfrak{g}} \cong \prod_{i \geq 1} \mathfrak{g}_i$. Therefore, the morphism $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ is injective, and induces an isomorphism $\mathfrak{g} \cong \text{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{g}})$. Moreover, if \mathfrak{h} is a graded Lie subalgebra of \mathfrak{g} , then $\widehat{\mathfrak{h}} = \mathfrak{h}$ and

$$(7) \quad \text{gr}^{\widehat{\Gamma}}(\widehat{\mathfrak{h}}) = \mathfrak{h}.$$

Lemma 2.3. *If \mathfrak{L} is a free Lie algebra generated in degree 1, and \mathfrak{r} is a homogeneous ideal, then the projection $\pi: \mathfrak{L} \rightarrow \mathfrak{L} / \mathfrak{r}$ induces an isomorphism $\widehat{\mathfrak{L}} / \widehat{\mathfrak{r}} \xrightarrow{\cong} \widehat{\mathfrak{L}} / \mathfrak{r}$.*

Proof. Without loss of generality, we may assume that $\mathfrak{r} \subset [\mathfrak{L}, \mathfrak{L}]$. The projection $\pi: \mathfrak{L} \rightarrow \mathfrak{L}/\mathfrak{r}$ extends to an epimorphism between the degree completions, $\hat{\pi}: \widehat{\mathfrak{L}} \rightarrow \widehat{\mathfrak{L}/\mathfrak{r}}$. This morphism takes the ideal generated by \mathfrak{r} to 0; thus, by continuity, it induces an epimorphism of complete, filtered Lie algebras, $\widehat{\mathfrak{L}/\mathfrak{r}} \twoheadrightarrow \widehat{\mathfrak{L}/\mathfrak{r}}$. Taking associated graded, we obtain an epimorphism $\text{gr}(\hat{\pi}): \text{gr}(\widehat{\mathfrak{L}/\mathfrak{r}}) \twoheadrightarrow \text{gr}(\mathfrak{L}/\mathfrak{r}) = \mathfrak{L}/\mathfrak{r}$. This epimorphism admits a splitting, induced by the maps $\Gamma_n \mathfrak{L} + \mathfrak{r} \rightarrow \Gamma_n \widehat{\mathfrak{L}} + \bar{\mathfrak{r}}$; thus, $\text{gr}(\hat{\pi})$ is an isomorphism. The claim now follows from Lemma 2.2. \square

2.6. Filtered formality. We now consider in more detail the relationship between a filtered Lie algebra \mathfrak{g} and the completion of its associated graded Lie algebra, $\widehat{\text{gr}}(\mathfrak{g})$, endowed with the inverse limit filtration. Note that both Lie algebras share the same associated graded Lie algebra, namely, $\text{gr}(\mathfrak{g})$. In general, though, \mathfrak{g} may not be isomorphic to $\widehat{\text{gr}}(\mathfrak{g})$. Of course, this happens if \mathfrak{g} is not complete or separated, but it may happen even in the case when \mathfrak{g} is a (finite-dimensional) nilpotent Lie algebra. We shall illustrate this point in Examples 10.5 and 10.6 below.

The following definition will play a key role in the sequel.

Definition 2.4. A complete, separated, filtered Lie algebra \mathfrak{g} is said to be *filtered-formal* if there is a filtered Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\text{gr}}(\mathfrak{g})$ which induces the identity on associated graded Lie algebras.

This notion appears in the work of Bezrukavnikov [7] and Hain [33], as well as in the work of Calaque–Enriquez–Etingof [10] under the name of ‘formality’, and in the work of Lee [47], under the name of ‘weak-formality’. The reasons for our choice of terminology will become more apparent in §6.

If \mathfrak{g} is a filtered-formal Lie algebra, there exists a graded Lie algebra \mathfrak{h} such that \mathfrak{g} is isomorphic to $\widehat{\mathfrak{h}} = \prod_{i \geq 1} \mathfrak{h}_i$. Conversely, if $\mathfrak{g} = \widehat{\mathfrak{h}}$ is the completion of a graded Lie algebra $\mathfrak{h} = \bigoplus_{i \geq 1} \mathfrak{h}_i$, then \mathfrak{g} is filtered-formal. Moreover, if \mathfrak{h} has homogeneous presentation $\mathfrak{h} = \text{lie}(V)/\mathfrak{r}$, with V finitely generated and concentrated in degree 1, then, by Lemma 2.3, the complete, filtered Lie algebra $\mathfrak{g} = \prod_{i \geq 1} \mathfrak{h}_i$ has presentation $\mathfrak{g} = \widehat{\text{lie}(V)}/\bar{\mathfrak{r}}$.

Lemma 2.5. Let \mathfrak{g} be a complete, separated, filtered Lie algebra. If there is a graded Lie algebra \mathfrak{h} and a Lie algebra isomorphism $\mathfrak{g} \cong \widehat{\mathfrak{h}}$ preserving filtrations, then \mathfrak{g} is filtered-formal.

Proof. By assumption, there exists a filtered Lie algebra isomorphism $\varphi: \mathfrak{g} \rightarrow \widehat{\mathfrak{h}}$. The map φ induces an isomorphism of graded Lie algebras, $\text{gr}(\varphi): \text{gr}(\mathfrak{g}) \rightarrow \mathfrak{h}$. In turn, the map $\psi := (\text{gr}(\varphi))^{-1}$ induces an isomorphism $\widehat{\psi}: \widehat{\mathfrak{h}} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$ of completed Lie algebras. Hence, the composite $\tilde{\varphi} := \widehat{\psi} \circ \varphi: \mathfrak{g} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$ is an isomorphism of filtered Lie algebras inducing the identity on $\text{gr}(\mathfrak{g})$. The conclusion follows from Lemma 2.2. \square

Corollary 2.6. Let \mathfrak{g} be a complete, separated, filtered Lie algebra, and suppose the associated graded Lie algebra $\text{gr}(\mathfrak{g})$ is finitely generated in degree 1. Furthermore, suppose there is a morphism of filtered Lie algebras, $\varphi: \mathfrak{g} \rightarrow \widehat{\text{gr}}(\mathfrak{g})$, such that $\text{gr}_1(\varphi)$ is an isomorphism. Then \mathfrak{g} is filtered-formal.

Proof. By Lemma 2.1, the map $\text{gr}(\varphi): \text{gr}(\mathfrak{g}) \rightarrow \text{gr}(\mathfrak{g})$ is an isomorphism. By Lemma 2.2, the map φ itself is an isomorphism. The conclusion follows from Lemma 2.5. \square

2.7. Descent of filtered formality. We now show that filtered-formality is compatible with extension of scalars, and, more importantly, that filtered formality enjoys a descent property, under some mild finiteness assumptions. As usual, all the ground fields will be of characteristic 0. We start with an easy lemma.

Lemma 2.7. *Let \mathfrak{g} be a filtered-formal \mathbb{k} -Lie algebra, and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Then the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is also filtered-formal.*

Proof. Follows from the fact that completion commutes with tensor products. \square

The proof of the next result is based on recent work of Enriquez [23] and Maassarani [52]. The key tool is Proposition 7.6 from [23], which in turn was inspired by work of Drinfeld [20]. The structure of the proof follows to a large extent the approach from [52], where a particular example (the Malcev Lie algebra of the fundamental group of the orbit configuration space of a finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ acting on \mathbb{CP}^1) is treated.

Theorem 2.8. *Let \mathfrak{g} be a complete, separated, filtered \mathbb{k} -Lie algebra such that $\mathrm{gr}(\mathfrak{g})$ is finitely generated in degree 1. If $\mathbb{k} \subset \mathbb{K}$ is a field extension, then \mathfrak{g} is filtered-formal if and only if the \mathbb{K} -Lie algebra $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal.*

Proof. The forward implication follows at once from Lemma 2.7. For the backward implication, suppose $\mathfrak{g} \otimes_{\mathbb{k}} \mathbb{K}$ is filtered-formal over \mathbb{K} . Again in view of Lemma 2.7, we may assume without loss of generality that $\mathbb{k} = \mathbb{Q}$ and $\mathbb{K} = \overline{\mathbb{K}}$.

Set $\mathfrak{h} = \widehat{\mathrm{gr}}(\mathfrak{g})$, and let $\{\mathcal{G}_k\}_{k \geq 1}$ be the inverse limit filtration on \mathfrak{h} , coming from the degree filtration on $\mathrm{gr}(\mathfrak{g})$. Clearly, $\mathrm{gr}(\mathfrak{h}) = \mathrm{gr}(\mathfrak{g})$. For simplicity, let us write $\mathfrak{g}^i = \mathfrak{g} / \mathcal{F}_{i+1}\mathfrak{g}$ and $\mathfrak{h}^i = \mathfrak{h} / \mathcal{G}_{i+1}\mathfrak{h}$ for the respective quotient Lie algebras. As noted in [52, Lem. 6.1], the image of $\mathcal{F}_k(\mathfrak{g})$ in \mathfrak{g}^i is $\Gamma_k(\mathfrak{g}^i)$. In particular, \mathfrak{g}^1 is canonically isomorphic to the abelianizations of all \mathfrak{g}^i and of \mathfrak{g} . A similar statement holds for \mathfrak{h} .

For each $i \geq 1$, let $T_i = \mathrm{Iso}_1(\mathfrak{g}^i, \mathfrak{h}^i)$ be the affine \mathbb{Q} -scheme of filtration-preserving Lie algebra isomorphisms from \mathfrak{g}^i to \mathfrak{h}^i inducing the identity on abelianizations. As shown in [52, Prop. 6.2], these schemes form in natural way an inverse system; let $T = \varprojlim_i T_i$. Similarly, let $U_i = \mathrm{Aut}_1(\mathfrak{g}^i)$ be the unipotent \mathbb{Q} -group of automorphisms of \mathfrak{g}^i inducing the identity on abelianization, and let $U = \varprojlim_i U_i$ be the corresponding prounipotent \mathbb{Q} -group scheme. It is then readily seen that each U_i is a torsor under the natural left action of T_i , i.e., the action of $U_i(\mathbf{k})$ on $T_i(\mathbf{k})$ is free and transitive whenever $\mathbb{Q} \subset \mathbf{k}$ is a field extension such that $T_i(\mathbf{k})$ is non-empty. Furthermore, as noted in [52, Prop. 6.6], the U_i -actions on the torsors T_i are compatible with the canonical projections; thus, T is also a torsor under the action of U .

By assumption, $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{K}$ is filtered-formal. In view of Corollary 2.6, this condition is equivalent to the existence of a filtered Lie algebra isomorphism $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{K} \rightarrow \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{K}$ inducing the canonical identification $\mathfrak{g}^1 \otimes_{\mathbb{Q}} \mathbb{K} = \mathfrak{h}^1 \otimes_{\mathbb{Q}} \mathbb{K}$. That is, our assumption is equivalent to the fact that $T(\mathbb{K}) \neq \emptyset$. It remains to show that $T(\mathbb{Q}) \neq \emptyset$.

In the case $\mathbb{K} = \mathbb{C}$, this claim follows at once from Proposition 7.6 in [23]. The proof of that proposition involves two steps: first a descent from \mathbb{C} to $\overline{\mathbb{Q}}$, and then a descent from $\overline{\mathbb{Q}}$ to \mathbb{Q} . To handle an arbitrary extension $\mathbb{Q} \subset \mathbb{K}$, we only need to modify the first step, and descend from $\mathbb{K} = \overline{\mathbb{K}}$ to $\overline{\mathbb{Q}}$. This is done by means of the same type of Hilbert's Nullstellensatz argument as the one sketched in [23]; we refer to the proof of Corollary 5.8 from [61] for more details on how such an argument works. \square

As we shall see in Corollary 10.3, the above theorem generalizes a recent result of Cornulier (Theorem 3.14 from [16]).

2.8. Products and coproducts. The category of Lie algebras admits both products and coproducts. We conclude this section by showing that filtered formality behaves well with respect to these operations.

Lemma 2.9. *Let \mathfrak{m} and \mathfrak{n} be two filtered-formal Lie algebras. Then $\mathfrak{m} \times \mathfrak{n}$ is also filtered-formal.*

Proof. By assumption, there exist graded Lie algebras \mathfrak{g} and \mathfrak{h} such that $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{i \geq 1} \mathfrak{g}_i$ and $\mathfrak{n} \cong \widehat{\mathfrak{h}} = \prod_{i \geq 1} \mathfrak{h}_i$. We then have

$$(8) \quad \mathfrak{m} \times \mathfrak{n} \cong \left(\prod_{i \geq 1} \mathfrak{g}_i \right) \times \left(\prod_{i \geq 1} \mathfrak{h}_i \right) = \prod_{i \geq 1} (\mathfrak{g}_i \times \mathfrak{h}_i) = \widehat{\mathfrak{g} \times \mathfrak{h}}.$$

Hence, $\mathfrak{m} \times \mathfrak{n}$ is filtered-formal. \square

Now let $*$ denote the usual coproduct (or, free product) of Lie algebras, and let $\hat{*}$ be the coproduct in the category of complete, filtered Lie algebras. By definition,

$$(9) \quad \mathfrak{m} \hat{*} \mathfrak{n} = \widehat{\mathfrak{m} * \mathfrak{n}} = \varprojlim_k (\mathfrak{m} * \mathfrak{n}) / \Gamma_k(\mathfrak{m} * \mathfrak{n}).$$

We refer to Lazarev and Markl [46] for a detailed study of this notion.

Lemma 2.10. *Let \mathfrak{m} and \mathfrak{n} be two filtered-formal Lie algebras. Then $\mathfrak{m} \hat{*} \mathfrak{n}$ is also filtered-formal.*

Proof. As before, write $\mathfrak{m} = \widehat{\mathfrak{g}}$ and $\mathfrak{n} = \widehat{\mathfrak{h}}$, for some graded Lie algebras \mathfrak{g} and \mathfrak{h} . The canonical inclusions, $\alpha: \mathfrak{g} \hookrightarrow \mathfrak{m}$ and $\beta: \mathfrak{h} \hookrightarrow \mathfrak{n}$, induce a monomorphism of filtered Lie algebras, $\widehat{\alpha * \beta}: \widehat{\mathfrak{g} * \mathfrak{h}} \rightarrow \widehat{\mathfrak{m} * \mathfrak{n}}$. Using [46, (9.3)], we infer that the induced morphism between associated graded Lie algebras, $\text{gr}(\widehat{\alpha * \beta}): \text{gr}(\widehat{\mathfrak{g} * \mathfrak{h}}) \rightarrow \text{gr}(\widehat{\mathfrak{m} * \mathfrak{n}})$, is an isomorphism. Lemma 2.2 now implies that $\widehat{\alpha * \beta}$ is an isomorphism of filtered Lie algebras, thereby verifying the filtered-formality of $\mathfrak{m} \hat{*} \mathfrak{n}$. \square

3. GRADED ALGEBRAS AND KOSZUL DUALITY

The notions of graded and filtered algebras are defined completely analogously for an (associative) algebra A : the multiplication map is required to preserve the grading, respectively the filtration on A . In this section we discuss several relationships between Lie algebras and associative algebras, focussing on the notion of quadratic and Koszul algebras.

3.1. Universal enveloping algebras. Given a Lie algebra \mathfrak{g} over a field \mathbb{k} of characteristic 0, let $U(\mathfrak{g})$ be its universal enveloping algebra. This is the filtered algebra obtained as the quotient of the tensor algebra $T(\mathfrak{g})$ by the (two-sided) ideal I generated by all elements of the form $a \otimes b - b \otimes a - [a, b]$ with $a, b \in \mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, the canonical map $\iota: \mathfrak{g} \rightarrow U(\mathfrak{g})$ is an injection, and the induced map, $\text{Sym}(\mathfrak{g}) \rightarrow \text{gr}(U(\mathfrak{g}))$, is an isomorphism of graded (commutative) algebras. In this section, all tensor products are over \mathbb{k} .

Now suppose \mathfrak{g} is a finitely generated, graded Lie algebra. Then $U(\mathfrak{g})$ is isomorphic (as a graded vector space) to a polynomial algebra in variables indexed by bases for the graded pieces of \mathfrak{g} , with degrees set accordingly. Hence, its Hilbert series is given by

$$(10) \quad \text{Hilb}(U(\mathfrak{g}), t) = \prod_{i \geq 1} (1 - t^i)^{-\dim(\mathfrak{g}_i)}.$$

For instance, if $\mathfrak{g} = \text{lie}(V)$ is the free Lie algebra on a finite-dimensional vector space V with all generators in degree 1, then $\dim(\mathfrak{g}_i) = \frac{1}{i} \sum_{d|i} \mu(d) \cdot n^{i/d}$, where $n = \dim V$ and $\mu: \mathbb{N} \rightarrow \{-1, 0, 1\}$ is the Möbius function.

Finally, suppose $\mathfrak{g} = \text{lie}(V)/\mathfrak{r}$ is a finitely presented, graded Lie algebra, with generators in degree 1 and relation ideal \mathfrak{r} generated by homogeneous elements g_1, \dots, g_m . Then $U(\mathfrak{g})$ is the quotient of $T(V)$ by the two-sided ideal generated by $\iota(g_1), \dots, \iota(g_m)$, where $\iota: \text{lie}(V) \hookrightarrow T(V)$ is the canonical inclusion. In particular, if \mathfrak{g} is a quadratic Lie algebra, then $U(\mathfrak{g})$ is a quadratic algebra.

3.2. Quadratic algebras. Now let A be a graded \mathbb{k} -algebra. We will assume throughout that A is non-negatively graded, i.e., $A = \bigoplus_{i \geq 0} A_i$, and connected, i.e., $A_0 = \mathbb{k}$. Every such algebra may be realized as the quotient of a tensor algebra $T(V)$ by a homogeneous, two-sided ideal I . We will further assume that $\dim V < \infty$.

An algebra A as above is said to be *quadratic* if $A_1 = V$ and the ideal I is generated in degree 2, i.e., $I = \langle I_2 \rangle$, where $I_2 = I \cap (V \otimes V)$. Given a quadratic algebra $A = T(V)/I$, identify $V^* \otimes V^* \cong (V \otimes V)^*$, and define the *quadratic dual* of A to be the algebra

$$(11) \quad A^\perp = T(V^*)/I^\perp,$$

where $I^\perp \subset T(V^*)$ is the ideal generated by the vector subspace $I_2^\perp := \{\alpha \in V^* \otimes V^* \mid \alpha(I_2) = 0\}$. Clearly, A^\perp is again a quadratic algebra, and $(A^\perp)^\perp = A$.

For any graded algebra $A = T(V)/I$, we can define a quadrature closure $\bar{A} = T(V)/\langle I_2 \rangle$.

Proposition 3.1. *Let \mathfrak{g} be a finitely generated graded Lie algebra generated in degree 1. There is then a unique, functorially defined quadratic Lie algebra, $\bar{\mathfrak{g}}$, such that $U(\bar{\mathfrak{g}}) = \overline{U(\mathfrak{g})}$.*

Proof. Suppose \mathfrak{g} has presentation $\text{lie}(V)/\mathfrak{r}$. Then $U(\mathfrak{g})$ has a presentation $T(V)/\langle \iota(\mathfrak{r}) \rangle$. Set $\bar{\mathfrak{g}} = \text{lie}(V)/\langle \mathfrak{r}_2 \rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \text{lie}_2(V)$; then $U(\bar{\mathfrak{g}})$ has presentation $T(V)/\langle \iota(\mathfrak{r}_2) \rangle$. One can see that $\iota(\mathfrak{r}_2) = \iota(\mathfrak{r}) \cap V \otimes V$. \square

A *commutative graded algebra* (for short, a cga) is a graded \mathbb{k} -algebra as above, which in addition is graded-commutative, i.e., if $a \in A_i$ and $b \in A_j$, then $ab = (-1)^{ij}ba$. If all generators of A are in degree 1, then A can be written as $A = \wedge(V)/J$, where $\wedge(V)$ is the exterior algebra on the \mathbb{k} -vector space $V = A_1$, and J is a homogeneous ideal in $\wedge(V)$ with $J_1 = 0$. If, furthermore, J is generated in degree 2, then A is a quadratic cga. The next lemma follows straight from the definitions.

Lemma 3.2. *Let $W \subset V \wedge V$ be a linear subspace, and let $A = \wedge(V)/\langle W \rangle$ be the corresponding quadratic cga. Then $A^\perp = T(V^*)/\langle \iota(W^\vee) \rangle$, where*

$$(12) \quad W^\vee := \{\alpha \in V^* \wedge V^* \mid \alpha(W) = 0\} = W^\perp \cap (V^* \wedge V^*),$$

and $\iota: V^* \wedge V^* \hookrightarrow V^* \otimes V^*$ is the inclusion map, given by $x \wedge y \mapsto x \otimes y - y \otimes x$.

For instance, if $A = \wedge(V)$, then $A^\perp = \text{Sym}(V^*)$. Likewise, if $A = \wedge(V)/\langle V \wedge V \rangle = \mathbb{k} \oplus V$, then $A^\perp = T(V^*)$.

3.3. Holonomy Lie algebras. Let A be a commutative graded algebra. Recall we are assuming that $A_0 = \mathbb{k}$ and $\dim A_1 < \infty$. Because of graded-commutativity, the multiplication map $A_1 \otimes A_1 \rightarrow A_2$ factors through a linear map $\mu_A: A_1 \wedge A_1 \rightarrow A_2$. Dualizing this map, and identifying $(A_1 \wedge A_1)^* \cong A_1^* \wedge A_1^*$, we obtain a linear map,

$$(13) \quad \partial_A = (\mu_A)^*: A_2^* \rightarrow A_1^* \wedge A_1^*.$$

Finally, identify $A_1^* \wedge A_1^*$ with $\text{lie}_2(A_1^*)$ via the map $x \wedge y \mapsto [x, y]$.

Definition 3.3. The *holonomy Lie algebra* of A is the quotient

$$(14) \quad \mathfrak{h}(A) = \text{lie}(A_1^*) / \langle \text{im } \partial_A \rangle$$

of the free Lie algebra on A_1^* by the ideal generated by the image of ∂_A under the above identification. Alternatively, using the notation from (12), we have that

$$(15) \quad \mathfrak{h}(A) = \text{lie}(A_1^*) / \langle \ker(\mu_A)^\vee \rangle.$$

By construction, $\mathfrak{h}(A)$ is a quadratic Lie algebra. Moreover, this construction is functorial: if $\varphi: A \rightarrow B$ is a morphism of cGAS as above, the induced map, $\text{lie}(\varphi_1^*): \text{lie}(B_1^*) \rightarrow \text{lie}(A_1^*)$, factors through a morphism of graded Lie algebras, $\mathfrak{h}(\varphi): \mathfrak{h}(B) \rightarrow \mathfrak{h}(A)$. Moreover, if φ is injective, then $\mathfrak{h}(\varphi)$ is surjective.

Clearly, the holonomy Lie algebra $\mathfrak{h}(A)$ depends only on the information encoded in the multiplication map $\mu_A: A_1 \wedge A_1 \rightarrow A_2$. More precisely, let \bar{A} be the *quadratic closure* of A , defined as

$$(16) \quad \bar{A} = \bigwedge (A_1) / \langle K \rangle,$$

where $K = \ker(\mu_A) \subset A_1 \wedge A_1$. Then \bar{A} is a commutative, quadratic algebra, which comes equipped with a canonical homomorphism $q: \bar{A} \rightarrow A$, which is an isomorphism in degree 1 and a monomorphism in degree 2. It is readily verified that the induced morphism between holonomy Lie algebras, $\mathfrak{h}(A) \rightarrow \mathfrak{h}(\bar{A})$, is an isomorphism.

The following proposition is a slight generalization of a result of Papadima–Yuzvinsky [67, Lemma 4.1].

Proposition 3.4. *Let A be a commutative graded algebra. Then $U(\mathfrak{h}(A))$ is a quadratic algebra, and $U(\mathfrak{h}(A)) = \bar{A}^!$.*

Proof. By the above, $\bar{A} = \bigwedge (A_1) / \langle K \rangle$, where $K = \langle \ker(\mu_A) \rangle$. On the other hand, by (15) we have that $\mathfrak{h}(A) = \text{lie}(A_1^*) / \langle K^\vee \rangle$. Hence, by Lemma 3.2, $U(\mathfrak{h}(A)) = T(V^*) / \langle u(K^\vee) \rangle = \bar{A}^!$. \square

Combining Propositions 3.1 and 3.4, we obtain the following corollary, which expresses the quadratic closure of a Lie algebra as the holonomy Lie algebra of a certain quadratic algebra.

Corollary 3.5. *Let \mathfrak{g} be a finitely generated graded Lie algebra generated in degree 1. Then $\mathfrak{h}(\overline{U(\mathfrak{g})}^!) = \bar{\mathfrak{g}}$.*

Work of Löfwall [51, Thm. 1.1] yields another interpretation of the universal enveloping algebra of the holonomy Lie algebra.

Proposition 3.6 ([51]). *Let $\text{Ext}_A^1(\mathbb{k}, \mathbb{k}) = \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$ be the linear strand in the Yoneda algebra of A . Then $U(\mathfrak{h}(A)) \cong \text{Ext}_A^1(\mathbb{k}, \mathbb{k})$.*

In particular, the graded ranks of the holonomy Lie algebra $\mathfrak{h} = \mathfrak{h}(A)$ are given by $\prod_{n \geq 1} (1 - t^n)^{\dim(\mathfrak{h}_n)} = \sum_{i \geq 0} b_{ii} t^i$, where $b_{ii} = \dim_{\mathbb{k}} \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$.

The next proposition shows that every quadratic Lie algebra can be realized as the holonomy Lie algebra of a (quadratic) algebra.

Proposition 3.7. *Let \mathfrak{g} be a quadratic Lie algebra. There is then a commutative quadratic algebra A such that $\mathfrak{g} = \mathfrak{h}(A)$.*

Proof. By assumption, \mathfrak{g} has a presentation of the form $\mathrm{lie}(V)/\langle W \rangle$, where W is a linear subspace of $V \wedge V$. Define $A = \wedge(V^*)/\langle W^\vee \rangle$. Then, by (15),

$$(17) \quad \mathfrak{h}(A) = \mathrm{lie}((V^*)^*)/\langle (W^\vee)^\vee \rangle = \mathrm{lie}(V)/\langle W \rangle,$$

and this completes the proof. \square

3.4. Koszul algebras. Any connected, graded algebra $A = \bigoplus_{i \geq 0} A_i$ has a free, graded A -resolution of the trivial A -module \mathbb{k} ,

$$(18) \quad \cdots \xrightarrow{\varphi_3} A^{b_2} \xrightarrow{\varphi_2} A^{b_1} \xrightarrow{\varphi_1} A \longrightarrow \mathbb{k}.$$

Such a resolution is said to be *minimal* if all the nonzero entries of the matrices φ_i have positive degrees.

A *Koszul algebra* is a graded algebra for which the minimal graded resolution of \mathbb{k} is linear, or, equivalently, $\mathrm{Ext}_A(\mathbb{k}, \mathbb{k}) = \mathrm{Ext}_A^1(\mathbb{k}, \mathbb{k})$. Such an algebra is always quadratic, but the converse is far from true. If A is a Koszul algebra, then the quadratic dual $A^!$ is also a Koszul algebra, and the following ‘Koszul duality’ formula holds:

$$(19) \quad \mathrm{Hilb}(A, t) \cdot \mathrm{Hilb}(A^!, -t) = 1.$$

Furthermore, if A is a graded algebra of the form $A = T(V)/I$, where I is an ideal admitting a (noncommutative) quadratic Gröbner basis, then A is a Koszul algebra (see [30] by Fröberg).

Corollary 3.8. *Let A be a connected, commutative graded algebra. If \bar{A} is a Koszul algebra, then $\mathrm{Hilb}(\bar{A}, -t) \cdot \mathrm{Hilb}(U(\mathfrak{h}(A)), t) = 1$.*

Example 3.9. Consider the quadratic algebra $A = \wedge(u_1, u_2, u_3, u_4)/(u_1u_2 - u_3u_4)$. Clearly, we have $\mathrm{Hilb}(A, t) = 1 + 4t + 5t^2$. If A were Koszul, then formula (19) would give $\mathrm{Hilb}(A^!, t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 - 29t^6 + \cdots$, which is impossible.

Example 3.10. The quasitriangular Lie algebra qtr_n defined in [2] is generated by x_{ij} , $1 \leq i \neq j \leq n$ with relations $[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0$ for distinct i, j, k and $[x_{ij}, x_{kl}] = 0$ for distinct i, j, k, l . The Lie algebra tr_n is the quotient Lie algebra of qtr_n by the ideal generated by $x_{ij} + x_{ji}$ for distinct $i \neq j$. In [2], Bartholdi et al. show that the quadratic dual algebras $U(\mathrm{qtr}_n)^!$ and $U(\mathrm{tr}_n)^!$ are Koszul, and compute their Hilbert series. They also state that neither qtr_n nor tr_n is filtered-formal for $n \geq 4$, and sketch a proof of this fact. We will provide a detailed proof in [79].

4. MINIMAL MODELS AND (PARTIAL) FORMALITY

In this section, we discuss two basic notions in non-simply-connected rational homotopy theory: the minimal model and the (partial) formality properties of a differential graded algebra.

4.1. Minimal models of dgas. We follow the approach of Sullivan [83], Deligne et al. [18], and Morgan [60], as further developed by Félix et al. [25, 26], Griffiths and Morgan [31], Halperin and Stasheff [35], Kohno [41], and Măcinic [53]. We start with some basic algebraic notions.

Definition 4.1. A *differential graded algebra* (for short, a *dga*) over a field \mathbb{k} of characteristic 0 is a graded \mathbb{k} -algebra $A^* = \bigoplus_{n \geq 0} A^n$ equipped with a differential $d: A \rightarrow A$ of degree 1 satisfying $ab = (-1)^{mn}ba$ and $d(ab) = d(a) \cdot b + (-1)^{|a|}a \cdot d(b)$ for any $a \in A^m$ and $b \in A^n$. We denote the dga by (A^*, d) or simply by A^* if there is no confusion.

A morphism $f: A^* \rightarrow B^*$ between two DGA's is a degree zero algebra map which commutes with the differentials. A *Hirsch extension* (of degree i) is a DGA inclusion $\alpha: (A^*, d_A) \hookrightarrow (A^* \otimes \wedge(V), d)$, where V is a \mathbb{k} -vector space concentrated in degree i , while $\wedge(V)$ is the free graded-commutative algebra generated by V , and d sends V into A^{i+1} . We say this is a *finite* Hirsch extension if $\dim V < \infty$. Note that all tensor product in this section is over \mathbb{k} , hence we will denote \otimes for short.

We now come to a crucial definition in rational homotopy theory, due to Sullivan [83].

Definition 4.2. A DGA (A^*, d) is called *minimal* if $A^0 = \mathbb{k}$, and the following two conditions are satisfied:

- (1) $A^* = \bigcup_{j \geq 0} A_j^*$, where $A_0 = \mathbb{k}$, and A_j is a Hirsch extension of A_{j-1} , for all $j \geq 0$.
- (2) The differential is *decomposable*, i.e., $dA^* \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{i \geq 1} A^i$.

The first condition implies that A^* has an increasing, exhausting filtration by the sub-DGA's A_j^* ; equivalently, A^* is free as a graded-commutative algebra on generators of degree ≥ 1 . (Note that we use the lower-index for the filtration, and the upper-index for the grading.) The second condition is automatically satisfied if A is generated in degree 1.

Two DGAs A^* and B^* are said to be *quasi-isomorphic* if there is a morphism $f: A \rightarrow B$ inducing isomorphisms in cohomology. The two DGAs are called *weakly equivalent* (written $A \simeq B$) if there is a sequence of quasi-isomorphisms (in either direction) connecting them. Likewise, for an integer $i \geq 0$, we say that a morphism $f: A \rightarrow B$ is an *i-quasi-isomorphism* if $f^*: H^j(A) \rightarrow H^j(B)$ is an isomorphism for each $j \leq i$ and $f^{i+1}: H^{i+1}(A) \rightarrow H^{i+1}(B)$ is injective. Furthermore, we say that A and B are *i-weakly equivalent* ($A \simeq_i B$) if there is a zig-zag of *i-quasi-isomorphism* connecting A to B .

The next two lemmas follow straight from the definitions.

Lemma 4.3. Any DGA morphism $\phi: (A, d_A) \rightarrow (B, d_B)$ extends to a DGA morphism of Hirsch extensions, $\bar{\phi}: (A, d_A) \otimes \wedge(x) \rightarrow (B, d_B) \otimes \wedge(y)$, provided that $d(y) = \phi(d(x))$. Moreover, if ϕ is a (quasi-) isomorphism, then so is $\bar{\phi}$.

Lemma 4.4. Let $\alpha: A \rightarrow B$ be the inclusion map of Hirsch extension of degree $i + 1$. Then α is an *i-quasi-isomorphism*.

Given a DGA A , we say that another DGA B is a *minimal model* for A if B is a minimal DGA and there exists a quasi-isomorphism $f: B \rightarrow A$. Likewise, we say that a minimal DGA B is an *i-minimal model* for A if B is generated by elements of degree at most i , and there exists an *i-quasi-isomorphism* $f: B \rightarrow A$. A basic result in rational homotopy theory is the following existence and uniqueness theorem, first proved for (full) minimal models by Sullivan [83], and in full generality by Morgan in [60, Thm. 5.6].

Theorem 4.5 ([60, 83]). *Each connected DGA (A, d) has a minimal model $\mathcal{M}(A)$, unique up to isomorphism. Likewise, for each $i \geq 0$, there is an *i-minimal model* $\mathcal{M}(A, i)$, unique up to isomorphism.*

It follows from the proof of Theorem 4.5 that the minimal model $\mathcal{M}(A)$ is isomorphic to a minimal model built from the *i-minimal model* $\mathcal{M}(A, i)$ by means of Hirsch extensions in degrees $i + 1$ and higher. Thus, in view of Lemma 4.4, $\mathcal{M}(A) \simeq_i \mathcal{M}(A, i)$.

4.2. Minimal models and holonomy Lie algebras. Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal DGA over \mathbb{k} , generated in degree 1. Following [60, 41], let us consider the filtration

$$(20) \quad \mathbb{k} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M} = \bigcup_i \mathcal{M}_i,$$

where \mathcal{M}_1 is the subalgebra of \mathcal{M} generated by $x \in \mathcal{M}^1$ such that $dx = 0$, and \mathcal{M}_i is the subalgebra of \mathcal{M} generated by $x \in \mathcal{M}^1$ such that $dx \in \mathcal{M}_{i-1}$ for $i > 1$. Each inclusion $\mathcal{M}_{i-1} \subset \mathcal{M}_i$ is a Hirsch extension of the form $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \wedge(V_i)$, where $V_i := \ker(H^2(\mathcal{M}_{i-1}) \rightarrow H^2(\mathcal{M}))$. Taking the degree 1 part of the filtration (20), we obtain the filtration

$$(21) \quad \mathbb{k} = \mathcal{M}_0^1 \subset \mathcal{M}_1^1 \subset \mathcal{M}_2^1 \subset \cdots \subset \mathcal{M}^1.$$

Now assume each of the above Hirsch extensions is finite, i.e., $\dim(V_i) < \infty$ for all i . Using the fact that $d(V_i) \subset \mathcal{M}_{i-1}$, we see that each dual vector space $\mathfrak{L}_i = (\mathcal{M}_i^1)^*$ acquires the structure of a \mathbb{k} -Lie algebra by setting

$$(22) \quad \langle [u^*, v^*], w \rangle = \langle u^* \wedge v^*, dw \rangle$$

for $v, w \in \mathcal{M}_i^1$. Clearly, $d(V_1) = 0$, and thus $\mathfrak{L}_1 = (V_1)^*$ is an abelian Lie algebra. Using the vector space decompositions $\mathcal{M}_i^1 = \mathcal{M}_{i-1}^1 \oplus V_i$ and $\mathcal{M}_i^2 = \mathcal{M}_{i-1}^2 \oplus (\mathcal{M}_{i-1}^1 \otimes V_i) \oplus \wedge^2(V_i)$ we easily see that the canonical projection $\mathfrak{L}_i \rightarrow \mathfrak{L}_{i-1}$ (i.e., the dual of the inclusion map $\mathcal{M}_{i-1} \hookrightarrow \mathcal{M}_i$) has kernel V_i^* , and this kernel is central inside \mathfrak{L}_i . Therefore, we obtain a tower of finite-dimensional nilpotent \mathbb{k} -Lie algebras,

$$(23) \quad 0 \longleftarrow \mathfrak{L}_1 \longleftarrow \mathfrak{L}_2 \longleftarrow \cdots \longleftarrow \mathfrak{L}_i \longleftarrow \cdots.$$

The inverse limit of this tower, $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$, endowed with the inverse limit filtration, is a complete, filtered Lie algebra with the property that $\mathfrak{L}/\widehat{\Gamma}_{i+1}\mathfrak{L} = \mathfrak{L}_i$, for each $i \geq 1$. Conversely, from a tower of the form (23), we can construct a sequence of finite Hirsch extensions \mathcal{M}_i as in (20). It is readily seen that the DGA \mathcal{M}_i , with differential defined by (22), coincides with the Chevalley–Eilenberg complex $(\wedge(\mathfrak{L}_i^*), d)$ associated to the finite-dimensional Lie algebra $\mathfrak{L}_i = \mathfrak{L}(\mathcal{M}_i)$, as in [37, Section VII]. In particular,

$$(24) \quad H^*(\mathcal{M}_i) \cong H^*(\mathfrak{L}_i; \mathbb{k}).$$

The direct limit of the above sequence of Hirsch extensions, $\mathcal{M} = \bigcup_i \mathcal{M}_i$, is a minimal \mathbb{k} -DGA generated in degree 1, which we denote by $\mathcal{M}(\mathfrak{L})$. We obtain in this fashion an adjoint correspondence that sends \mathcal{M} to the pronilpotent Lie algebra $\mathfrak{L}(\mathcal{M})$ and conversely, sends a pronilpotent Lie algebra \mathfrak{L} to the minimal algebra $\mathcal{M}(\mathfrak{L})$. Under this correspondence, filtration-preserving DGA morphisms $\mathcal{M} \rightarrow \mathcal{N}$ get sent to filtration-preserving Lie morphisms $\mathfrak{L}(\mathcal{N}) \rightarrow \mathfrak{L}(\mathcal{M})$, and vice-versa.

4.3. Positive weights. Following Body et al. [9], Morgan [60], and Sullivan [83], we say that a CGA A^* has *positive weights* if each graded piece has a vector space decomposition $A^i = \bigoplus_{\alpha \in \mathbb{Z}} A^{i,\alpha}$ with $A^{1,\alpha} = 0$ for $\alpha \leq 0$, such that $xy \in A^{i+j,\alpha+\beta}$ for $x \in A^{i,\alpha}$ and $y \in A^{j,\beta}$. Furthermore, we say that a DGA (A^*, d) has *positive weights* if the underlying CGA A^* has positive weights, and the differential is homogeneous with respect to those weights, i.e., $d(x) \in A^{i+1,\alpha}$ for $x \in A^{i,\alpha}$.

Now let (\mathcal{M}^*, d) be a minimal DGA generated in degree one, endowed with the canonical filtration $\{\mathcal{M}_i\}_{i \geq 0}$ constructed in (20), where each sub-DGA \mathcal{M}_i given by a Hirsch extension of the form $\mathcal{M}_{i-1} \otimes \wedge(V_i)$. The underlying CGA \mathcal{M}^* possesses a natural set of positive weights, which we will refer to as the *Hirsch weights*: simply declare V_i to have weight i , and extend those weights to \mathcal{M}^*

multiplicatively. We say that the dGA (\mathcal{M}^*, d) has *positive Hirsch weights* if the differential d is homogeneous with respect to those weights. If this is the case, each sub-dGA \mathcal{M}_i also has positive Hirsch weights.

Lemma 4.6. *Let $\mathcal{M} = (\mathcal{M}^*, d)$ be a minimal dGA generated in degree one, with dual Lie algebra \mathfrak{L} . Then \mathcal{M} has positive Hirsch weights if and only if $\mathfrak{L} = \widehat{\text{gr}}(\mathfrak{L})$.*

Proof. As usual, write $\mathcal{M} = \bigcup \mathcal{M}_i$, with $\mathcal{M}_i = \mathcal{M}_{i-1} \otimes \wedge(V_i)$. Since \mathcal{M} is generated in degree one, the differential is homogeneous with respect to the Hirsch weights if and only if $d(V_s) \subset \bigoplus_{i+j=s} V_i \wedge V_j$, for all $s \geq 1$. Passing now to the dual Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ and using formula (22), we see that this condition is equivalent to having $[V_i^*, V_j^*] \subset V_{i+j}^*$, for all $i, j \geq 1$. In turn, this is equivalent to saying that each Lie algebra \mathfrak{L}_i is a graded Lie algebra with $\text{gr}_k(\mathfrak{L}_i) = V_k^*$, for each $k \leq i$, which means that the filtered Lie algebra $\mathfrak{L} = \varprojlim_i \mathfrak{L}_i$ coincides with the completion of its associated graded Lie algebra, $\widehat{\text{gr}}(\mathfrak{L})$. \square

Remark 4.7. The property that the differential of \mathcal{M} be homogeneous with respect to the Hirsch weights is stronger than saying that the Lie algebra $\mathfrak{L} = \mathfrak{L}(\mathcal{M})$ is filtered-formal. The fact that this can happen is illustrated in Example 10.2.

Remark 4.8. If a minimal dGA is generated in degree 1 and has positive weights, but these weights do not coincide with the Hirsch weights, then the dual Lie algebra need not be filtered-formal. This phenomenon is illustrated in Example 10.5: there is a finitely generated nilpotent Lie algebra \mathfrak{m} for which the Chevalley–Eilenberg complex $\mathcal{M}(\mathfrak{m}) = \wedge(\mathfrak{m}^*)$ has positive weights, but those weights are not the Hirsch weights; moreover, \mathfrak{m} is not filtered-formal.

4.4. Dual Lie algebra and holonomy Lie algebra. Let (B^*, d) be a dGA, and let $A = H^*(B)$ be its cohomology algebra. Assume A is connected and $\dim A^1 < \infty$, and let $\mu: A^1 \wedge A^1 \rightarrow A^2$ be the multiplication map. By the discussion from §4.1, there is a 1-minimal model $\mathcal{M}(B, 1)$ for our dGA, unique up to isomorphism.

A concrete way to build such a model can be found in [18, 31, 60]. The first two steps of this construction are easy to describe. Set $V_1 = A^1$ and define $\mathcal{M}(B, 1)_1 = \wedge(V_1)$, with differential $d = 0$. Next, set $V_2 = \ker(\mu)$ and define $\mathcal{M}(B, 1)_2 = \wedge(V_1 \oplus V_2)$, with $d|_{V_2}$ equal to the inclusion map $V_2 \hookrightarrow A^1 \wedge A^1$.

Let $\mathfrak{L}(B) = \mathfrak{L}(\mathcal{M}(B, 1))$ be the Lie algebra corresponding to the 1-minimal model of B . The next proposition, which generalizes a result of Kohno ([41, Lemma 4.9]), relates this Lie algebra to the holonomy Lie algebra $\mathfrak{h}(A)$ from Definition 3.3.

Proposition 4.9. *Let $\phi: \mathbf{L} \rightarrow \mathfrak{L}(B)$ be the morphism defined by extending the identity map of V_1^* to the free Lie algebra $\mathbf{L} = \text{lie}(V_1^*)$, and let $J = \ker(\phi)$. There exists then an isomorphism of graded Lie algebras, $\mathfrak{h}(A) \cong \mathbf{L}/\langle J \cap \mathbf{L}_2 \rangle$, where $\mathfrak{h}(A)$ is the holonomy Lie algebra of $A = H^*(B)$.*

Proof. Let $\text{gr}(\phi): \mathbf{L} \rightarrow \widehat{\text{gr}}(\mathfrak{L}(B))$ be the associated graded morphism of ϕ . Then the first graded piece $\text{gr}_1(\phi): V_1^* \rightarrow V_1^*$ is the identity, while the second graded piece $\text{gr}_2(\phi)$ can be identified with the Lie bracket map $V_1^* \wedge V_1^* \rightarrow V_2^*$, which is the dual of the differential $d: V_2 \rightarrow V_1 \wedge V_1$. From the construction of $\mathcal{M}(B, 1)_2$, there is an isomorphism $\ker d^* \cong \text{im } \mu^*$. Since $J \cap \mathbf{L}_2 = \ker(\text{gr}_2(\phi))$, we have that $\text{im } \mu^* = J \cap \mathbf{L}_2$, and the claim follows. \square

4.5. The completion of the holonomy Lie algebra. Let A^* be a commutative graded \mathbb{k} -algebra with $A^0 = \mathbb{k}$. Proceeding as above, by taking $B = A$ and $d = 0$ so that $H^*(B) = A$, we can construct

a 1-minimal model $\mathcal{M} = \mathcal{M}(A, 1)$ for the algebra A in a ‘formal’ way, following the approach outlined by Carlson and Toledo in [11]. (A construction of the full, bigraded minimal model of a CGA can be found in [35, §3].)

As before, set $\mathcal{M}_1 = (\wedge(V_1), d = 0)$ where $V_1 = A^1$, and $\mathcal{M}_2 = (\wedge(V_1 \oplus V_2), d)$, where $V_2 = \ker(\mu: A^1 \wedge A^1 \rightarrow A^2)$ and $d: V_2 \hookrightarrow V_1 \wedge V_1$ is the inclusion map. After that, define inductively \mathcal{M}_i as $\mathcal{M}_{i-1} \otimes \wedge(V_i)$, where the vector space V_i fits into the short exact sequence

$$(25) \quad 0 \longrightarrow V_i \longrightarrow H^2(\mathcal{M}_{i-1}) \longrightarrow \text{im}(\mu) \longrightarrow 0,$$

while the differential d includes V_i into $V_1 \wedge V_{i-1} \subset \mathcal{M}_{i-1}$. In particular, the subalgebras \mathcal{M}_i constitute the canonical filtration (20) of \mathcal{M} , and the differential d preserves the Hirsch weights on \mathcal{M} . For these reasons, we call $\mathcal{M} = \mathcal{M}(A, 1)$ the *canonical* 1-minimal model of A .

The next theorem relates the Lie algebra dual to the canonical 1-minimal model of a CGA as above to its holonomy Lie algebra. A similar result was obtained by Markl and Papadima in [56]; see also Morgan [60, Thm. 9.4] and Remark 7.3.

Theorem 4.10. *Let A^* be a connected CGA with $\dim A^1 < \infty$. Let $\mathfrak{L}(A) := \mathfrak{L}(\mathcal{M}(A, 1))$ be the Lie algebra corresponding to the canonical 1-minimal model of A , and let $\mathfrak{h}(A)$ be the holonomy Lie algebra of A . There exists then an isomorphism of complete, filtered Lie algebras between $\mathfrak{L}(A)$ and the degree completion $\widehat{\mathfrak{h}}(A)$.*

Proof. By Definition 3.3, the holonomy Lie algebra of A has presentation $\mathfrak{h}(A) = \mathbf{L}/\mathfrak{r}$, where $\mathbf{L} = \text{lie}(V_1^*)$ and \mathfrak{r} is the ideal generated by $\text{im}(\mu^*) \subset \mathbf{L}_2$. It follows that, for each $i \geq 1$, the nilpotent quotient $\mathfrak{h}_i(A) := \mathfrak{h}(A)/\Gamma_{i+1}\mathfrak{h}(A)$ has presentation $\mathbf{L}/(\mathfrak{r} + \Gamma_{i+1}\mathbf{L})$.

Consider now the dual Lie algebra $\mathfrak{L}_i(A) = \mathfrak{L}(\mathcal{M}_i)$. By construction, we have a vector space decomposition, $\mathfrak{L}_i(A) = \bigoplus_{s \leq i} V_s^*$. The fact that $d(V_s) \subset V_1 \wedge V_{s-1}$ implies that the Lie bracket maps $V_1^* \wedge V_{s-1}^*$ onto V_s^* , for every $1 < s \leq i$. In turn, this implies that $\mathfrak{L}_i(A)$ is an i -step nilpotent, graded Lie algebra generated in degree 1, with $\text{gr}_s(\mathfrak{L}_i(A)) = V_s^*$ for $s \leq i$.

Let \mathfrak{r}_i be the kernel of the canonical projection $\pi_i: \mathbf{L} \twoheadrightarrow \mathfrak{L}_i(A)$. By the Hopf formula, there is an isomorphism of graded vector spaces between $H_2(\mathfrak{L}_i(A); \mathbb{k})$ and $\mathfrak{r}_i/[\mathbf{L}, \mathfrak{r}_i]$, the space of (minimal) generators for the homogeneous ideal \mathfrak{r}_i . On the other hand, $H^2(\mathcal{M}_i) \cong H^2(\mathfrak{L}_i; \mathbb{k})$, by (24). Taking the dual of the exact sequence (25), we find that $H_2(\mathfrak{L}_i(A); \mathbb{k}) \cong \text{im}(\mu^*) \oplus V_{i+1}^*$. We conclude that the ideal \mathfrak{r}_i is generated by $\text{im}(\mu^*)$ in degree 2 and a copy of V_{i+1}^* in degree $i + 1$.

Since $\text{gr}_2(\mathfrak{r}) = \text{im}(\mu^*)$, we infer that $\bigoplus_{s \leq i} \text{gr}_s(\mathfrak{r}_i) = \bigoplus_{s \leq i} \text{gr}_s(\mathfrak{r})$. Since $\mathfrak{L}_i(A)$ is an i -step nilpotent Lie algebra, $\bigoplus_{s > i} \text{gr}_s(\mathfrak{r}_i) = \Gamma_{i+1}\mathbf{L}$. Therefore, $\Gamma_{i+1}\mathbf{L} + \mathfrak{r} = \mathfrak{r}_i$. It follows that the identity map of \mathbf{L} induces an isomorphism $\mathfrak{L}_i(A) \cong \mathfrak{h}_i(A)$, for each $i \geq 1$. Hence, $\mathfrak{L}(A) \cong \widehat{\mathfrak{h}}(A)$, as filtered Lie algebras. \square

Corollary 4.11. *The graded ranks of the holonomy Lie algebra of a connected, graded algebra A are given by $\dim \mathfrak{h}_i(A) = \dim V_i$, where $\mathcal{M} = \wedge(\bigoplus_{i \geq 1} V_i)$ is the canonical 1-minimal model of $(A, d = 0)$.*

4.6. Partial formality and field extensions. The following notion, introduced by Sullivan in [83], and further developed in [18, 31, 53, 60], will play a central role in our study.

Definition 4.12. A DGA (A^*, d) over \mathbb{k} is said to be *formal* if there exists a quasi-isomorphism $\mathcal{M}(A) \rightarrow (H^*(A), d = 0)$. Likewise, (A^*, d) is said to be *i -formal* if there exists an i -quasi-isomorphism $\mathcal{M}(A, i) \rightarrow (H^*(A), d = 0)$.

In [53], Măcinic studies in detail these concepts. Evidently, if A is formal, then it is i -formal, for all $i \geq 0$, and, if A is i -formal, then it is j -formal for every $j \leq i$. Moreover, A is 0-formal if and only if $H^0(A) = \mathbb{k}$.

Lemma 4.13 ([53]). *A DGA (A^*, d) is i -formal if and only if (A^*, d) is i -weakly equivalent to $H^*(A)$ with zero differential.*

As a corollary, we deduce that i -formality is invariant under i -weakly equivalence.

Corollary 4.14. *Suppose $A \simeq_i B$. Then A is i -formal if and only if B is i -formal.*

Given a DGA (A, d) over a field \mathbb{k} of characteristic 0, and a field extension $\mathbb{k} \subset \mathbb{K}$, let $(A \otimes \mathbb{K}, d \otimes \text{id}_{\mathbb{K}})$ be the corresponding DGA over \mathbb{K} . (If the underlying field \mathbb{k} is understood, we will usually omit it from the tensor product $A \otimes \mathbb{K}$.) The following result will be crucial for us in the sequel.

Theorem 4.15 (Thm. 6.8 in [35]). *Let (A^*, d_A) and (B^*, d_B) be two DGAs over \mathbb{k} whose cohomology algebras are connected and of finite type. Suppose there is an isomorphism of graded algebras, $f: H^*(A) \rightarrow H^*(B)$, and suppose $f \otimes \text{id}_{\mathbb{K}}: H^*(A) \otimes \mathbb{K} \rightarrow H^*(B) \otimes \mathbb{K}$ can be realized by a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ and $(B^* \otimes \mathbb{K}, d_B \otimes \text{id}_{\mathbb{K}})$. Then f can be realized by a weak equivalence between (A^*, d_A) and (B^*, d_B) .*

This theorem has an important corollary, based on the following lemma. For completeness, we provide proofs for these results, which are stated without proof in [35].

Lemma 4.16 ([35]). *A DGA (A^*, d_A) with $H^*(A)$ of finite-type is formal if and only if the identity map of $H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$.*

Proof. The backwards implication is obvious. So assume (A^*, d_A) is formal, that is, there is a zig-zag of quasi-isomorphisms between (A^*, d_A) and $(H^*(A), d = 0)$. This yields an isomorphism in cohomology, $\phi: H^*(A) \rightarrow H^*(A)$. The inverse of ϕ defines a quasi-isomorphism between $(H^*(A), d = 0)$ and $(H^*(A), d = 0)$. Composing this quasi-isomorphism with the given zig-zag of quasi-isomorphisms defines a new weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$, which induces the identity map in cohomology. \square

Corollary 4.17 ([35]). *A \mathbb{k} -DGA (A^*, d_A) with $H^*(A)$ of finite-type is formal if and only if the \mathbb{K} -DGA $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ is formal.*

Proof. As the forward implication is obvious, we only prove the converse. Suppose our \mathbb{K} -DGA is formal. By Lemma 4.16, there exists a weak equivalence between $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ and $(H^*(A) \otimes \mathbb{K}, d = 0)$ inducing the identity on $H^*(A) \otimes \mathbb{K}$. By Theorem 4.15, the map $\text{id}: H^*(A) \rightarrow H^*(A)$ can be realized by a weak equivalence between (A^*, d_A) and $(H^*(A), d = 0)$. That is, (A^*, d_A) is formal (over \mathbb{k}). \square

4.7. Field extensions and i -formality. We now use the aforementioned result of Halperin and Stasheff on full formality to establish an analogous result for partial formality. First we need an auxiliary construction, and a lemma.

Let $\mathcal{M}(A, i)$ be the i -minimal model of a DGA (A^*, d_A) . The degree $i + 1$ piece, $\mathcal{M}(A, i)^{i+1}$, is isomorphic to $(\ker d^{i+1}) \oplus \mathcal{C}_{i+1}$, where $d^{i+1}: \mathcal{M}(A, i)^{i+1} \rightarrow \mathcal{M}(A, i)^{i+2}$ is the differential, and \mathcal{C}_{i+1} is a complement to its kernel. It is readily checked that the vector subspace

$$(26) \quad \mathcal{I}_i := \mathcal{C}_{i+1} \oplus \bigoplus_{s \geq i+2} \mathcal{M}(A, i)^s$$

is an ideal of $\mathcal{M}(A, i)$, left invariant by the differential. Consider the quotient DGA,

$$(27) \quad \begin{aligned} \mathcal{M}[A, i] &:= \mathcal{M}(A, i) / \mathcal{I}_i \\ &= \mathbb{k} \oplus \mathcal{M}(A, i)^1 \oplus \cdots \oplus \mathcal{M}(A, i)^i \oplus \ker d^{i+1}. \end{aligned}$$

Lemma 4.18. *Suppose $\dim H^{i+1}(\mathcal{M}(A, i)) < \infty$. The following statements are then equivalent:*

- (1) (A^*, d_A) is i -formal.
- (2) $\mathcal{M}(A, i)$ is i -formal.
- (3) $\mathcal{M}[A, i]$ is i -formal.
- (4) $\mathcal{M}[A, i]$ is formal.

Proof. Since $\mathcal{M}(A, i)$ is an i -minimal model for (A^*, d_A) , the two DGAs are i -quasi-isomorphic. The equivalence (1) \Leftrightarrow (2) follows from Corollary 4.14.

Now let $\psi: \mathcal{M}(A, i) \rightarrow \mathcal{M}[A, i]$ be the canonical projection. It is readily checked that the induced homomorphism, $\psi^*: H^*(\mathcal{M}(A, i)) \rightarrow H^*(\mathcal{M}[A, i])$, is an isomorphism in degrees up to and including $i + 1$. In particular, this shows that $\mathcal{M}(A, i)$ is an i -minimal model for $\mathcal{M}[A, i]$. The equivalence (2) \Leftrightarrow (3) again follows from Corollary 4.14.

Implication (4) \Rightarrow (3) is trivial, so it remains to establish (3) \Rightarrow (4). Assume the DGA $\mathcal{M}[A, i]$ is i -formal. Since $\mathcal{M}(A, i)$ is an i -minimal model for $\mathcal{M}[A, i]$, there is an i -quasi-isomorphism β as in diagram (28). In particular, the homomorphism, $\beta^*: H^{i+1}(\mathcal{M}(A, i)) \rightarrow H^{i+1}(\mathcal{M}[A, i])$, is injective. On the other hand, we know from the previous paragraph that $H^{i+1}(\mathcal{M}[A, i])$ and $H^{i+1}(\mathcal{M}(A, i))$ have the same dimension. Since by assumption $\dim H^{i+1}(\mathcal{M}(A, i)) < \infty$, we conclude that β^* is an isomorphism in degree $i + 1$, too.

$$(28) \quad \begin{array}{ccc} \mathcal{M}(A, i) & \xrightarrow{\beta} & (H^*(\mathcal{M}[A, i]), 0) \\ \downarrow \psi & \searrow \alpha & \uparrow \gamma \simeq \\ \mathcal{M}[A, i] & \xleftarrow[\simeq]{\phi} & \mathcal{M} \end{array}$$

Let $\mathcal{M} = \mathcal{M}(\mathcal{M}[A, i])$ be the full minimal model of $\mathcal{M}[A, i]$. As mentioned right after Theorem 4.5, this model can be constructed by Hirsch extensions of degree $k \geq i + 1$, starting from the i -minimal model of $\mathcal{M}[A, i]$, which we can take to be $\mathcal{M}(A, i)$. Hence, the inclusion map, $\alpha: \mathcal{M}(A, i) \rightarrow \mathcal{M}$, induces isomorphisms in cohomology up to degree i , and a monomorphism in degree $i + 1$. Now, since $H^{i+1}(\mathcal{M})$ has the same dimension as $H^{i+1}(\mathcal{M}[A, i])$, and thus as $H^{i+1}(\mathcal{M}(A, i))$, the map α^* is also an isomorphism in degree $i + 1$.

The DGA morphism β extends to a CGA map $\gamma: \mathcal{M} \rightarrow H^*(\mathcal{M}[A, i])$ as in diagram (28), by sending the new generators to zero. Since the target of β vanishes in degrees $k \geq i + 2$ and has differential $d = 0$, the map γ is a DGA morphism. Furthermore, since $\gamma \circ \alpha = \beta$, we infer that γ induces isomorphisms in cohomology in degrees $k \leq i + 1$. Since $H^k(\mathcal{M}) = H^k(\mathcal{M}[A, i]) = 0$ for $k \geq i + 2$, we conclude that γ^* is an isomorphism in all degrees, i.e., γ is a quasi-isomorphism.

Finally, let $\phi: \mathcal{M} \rightarrow \mathcal{M}[A, i]$ be a quasi-isomorphism from the minimal model of $\mathcal{M}[A, i]$ to this DGA. The maps ϕ and γ define a weak equivalence between $\mathcal{M}[A, i]$ and $(H^*(\mathcal{M}[A, i]), 0)$, thereby showing that $\mathcal{M}[A, i]$ is formal. \square

Since $H^{\geq i+2}(\mathcal{M}[A, i]) = 0$, the equivalence of conditions (3) and (4) in the above lemma also follows from the (quite different) proof of Proposition 3.4 from [53]; see Remark 4.21 for more on this. We are now ready to prove descent for partial formality of DGAs.

Theorem 4.19. *Let (A^*, d_A) be a DGA over \mathbb{k} , and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Suppose $H^{\leq i+1}(A)$ is finite-dimensional and $H^0(A) = \mathbb{k}$. Then (A^*, d_A) is i -formal if and only if $(A^* \otimes \mathbb{K}, d_A \otimes \text{id}_{\mathbb{K}})$ is i -formal.*

Proof. From the definition of i -minimal model $\mathcal{M}(A, i)$ and the hypothesis that $H^{\leq i+1}(A)$ is finite-dimensional, we obtain that $H^{\leq i+1}(\mathcal{M}(A, i))$ is finite-dimensional. By Lemma 4.18, the DGA (A^*, d_A) is i -formal if and only if $\mathcal{M}[A, i]$ is formal. By construction, $H^q(\mathcal{M}[A, i])$ equals $H^q(A)$ for $q \leq i$, injects into $H^q(A)$ for $q = i + 1$, and vanishes for $q > i + 1$; hence, in view of our hypothesis, $H^*(\mathcal{M}[A, i])$ is of finite-type. By Corollary 4.17, $\mathcal{M}[A, i]$ is formal if and only if $\mathcal{M}[A, i] \otimes \mathbb{K}$ is formal. By Lemma 4.18 again, this is equivalent to the i -formality of $\mathcal{M}[A, i] \otimes \mathbb{K}$. \square

4.8. Formality notions for spaces. To every space X , Sullivan [83] associated in a functorial way a DGA of ‘rational polynomial forms’, denoted $A_{PL}^*(X)$. As shown in [25, §10], there is a natural identification $H^*(A_{PL}^*(X)) = H^*(X, \mathbb{Q})$ under which the respective induced homomorphisms in cohomology correspond. In particular, the weak isomorphism type of $A_{PL}^*(X)$ depends only on the rational homotopy type of X .

A DGA (A, d) over \mathbb{k} is called a *model* for the space X if A is weakly equivalent to Sullivan’s algebra $A_{PL}(X; \mathbb{k}) := A_{PL}(X) \otimes_{\mathbb{Q}} \mathbb{k}$. In other words, $\mathcal{M}(A)$ is isomorphic to $\mathcal{M}(X; \mathbb{k}) := \mathcal{M}(X) \otimes_{\mathbb{Q}} \mathbb{k}$, where $\mathcal{M}(A)$ is the minimal model of A and $\mathcal{M}(X)$ is the minimal model of $A_{PL}(X)$. In the same vein, A is an i -model for X if $(A, d) \simeq_i A_{PL}(X; \mathbb{k})$. For instance, if X is a smooth manifold, then the de Rham algebra $\Omega_{dR}^*(X)$ is a model for X over \mathbb{R} .

A space X is said to be *formal* over \mathbb{k} if the model $A_{PL}(X; \mathbb{k})$ is formal, that is, there is a quasi-isomorphism $\mathcal{M}(X; \mathbb{k}) \rightarrow (H^*(X; \mathbb{k}), d = 0)$. Likewise, X is said to be *i -formal*, for some $i \geq 0$, if there is an i -quasi-isomorphism $\mathcal{M}(A_{PL}(X; \mathbb{k}), i) \rightarrow (H^*(X; \mathbb{k}), d = 0)$. Note that X is 0-formal if and only if X is path-connected. Also, since a homotopy equivalence $X \simeq Y$ induces an isomorphism $H^*(Y; \mathbb{Q}) \xrightarrow{\cong} H^*(X; \mathbb{Q})$, it follows from Corollary 4.14 that the i -formality property is preserved under homotopy equivalences.

The following theorem of Papadima and Yuzvinsky [67] nicely relates the properties of the minimal model of X to the Koszulness of its cohomology algebra.

Theorem 4.20 ([67]). *Let X be a connected space with finite Betti numbers. If $\mathcal{M}(X) \cong \mathcal{M}(X, 1)$, then $H^*(X; \mathbb{Q})$ is a Koszul algebra. Moreover, if X is formal, then the converse holds.*

Remark 4.21. In [53, Prop. 3.4], Măcinic shows that every i -formal space X for which $H^{\geq i+2}(X; \mathbb{Q})$ vanishes is formal. In particular, the notions of formality and i -formality coincide for $(i + 1)$ -dimensional CW-complexes. In general, though, full formality is a much stronger condition than partial formality.

Remark 4.22. There is a competing notion of i -formality, due to Fernández and Muñoz [28]. As explained in [53], the two notions differ significantly, even for $i = 1$. In the sequel, we will use exclusively the classical notion of i -formality given above.

As is well-known, the (full) formality property behaves well with respect to field extensions of the form $\mathbb{Q} \subset \mathbb{k}$. Indeed, it follows from Halperin and Stasheff’s Corollary 4.17 that a connected space X with finite Betti numbers is formal over \mathbb{Q} if and only if X is formal over \mathbb{k} . This result was first stated and proved by Sullivan [83], using different techniques. An independent proof was given by Neisendorfer and Miller [61] in the simply-connected case.

These classical results on descent of formality may be strengthened to a result on descent of partial formality. More precisely, using Theorem 4.19, we obtain the following immediate corollary.

Corollary 4.23. *Let X be a connected space with finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i -formal over \mathbb{Q} if and only if X is i -formal over \mathbb{k} .*

5. GROUPS, LIE ALGEBRAS, AND GRADED FORMALITY

We now turn to finitely generated groups, and to two graded Lie algebras attached to such groups, with special emphasis on the relationship between these Lie algebras, leading to the notion of graded formality.

5.1. Central filtrations on groups. We start with some general background on lower central series and the associated graded Lie algebra of a group. For more details on this classical topic, we refer to Lazard [45] and Magnus et al. [54].

Let G be a group. For elements $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$ be their group commutator. Likewise, for subgroups $H, K < G$, let $[H, K]$ be the subgroup of G generated by all commutators $[x, y]$ with $x \in H, y \in K$.

A (central) filtration on the group G is a decreasing sequence of subgroups, $G = \mathcal{F}_1 G > \mathcal{F}_2 G > \mathcal{F}_3 G > \dots$, such that $[\mathcal{F}_r G, \mathcal{F}_s G] \subset \mathcal{F}_{r+s} G$. It is readily verified that, for each $k > 1$, the group $\mathcal{F}_{k+1} G$ is a normal subgroup of $\mathcal{F}_k G$, and the quotient group $\text{gr}_k^{\mathcal{F}}(G) = \mathcal{F}_k G / \mathcal{F}_{k+1} G$ is abelian. As before, let \mathbb{k} be a field of characteristic 0. The direct sum

$$(29) \quad \text{gr}^{\mathcal{F}}(G; \mathbb{k}) = \bigoplus_{k \geq 1} \text{gr}_k^{\mathcal{F}}(G) \otimes_{\mathbb{Z}} \mathbb{k}$$

is a graded Lie algebra over \mathbb{k} , with Lie bracket induced from the group commutator: If $x \in \mathcal{F}_r G$ and $y \in \mathcal{F}_s G$, then $[x + \mathcal{F}_{r+1} G, y + \mathcal{F}_{s+1} G] = xyx^{-1}y^{-1} + \mathcal{F}_{r+s+1} G$. We can view $\text{gr}^{\mathcal{F}}(-; \mathbb{k})$ as a functor from groups to graded \mathbb{k} -Lie algebras. Moreover, $\text{gr}^{\mathcal{F}}(G; \mathbb{K}) = \text{gr}^{\mathcal{F}}(G; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{K}$, for any field extension $\mathbb{k} \subset \mathbb{K}$. (Once again, if the underlying ring in a tensor product is understood, we will write \otimes for short.)

Let H be a normal subgroup of G , and let $Q = G/H$ be the quotient group. Define filtrations on H and Q by $\widetilde{\mathcal{F}}_k H = \mathcal{F}_k G \cap H$ and $\widetilde{\mathcal{F}}_k Q = \mathcal{F}_k G / \widetilde{\mathcal{F}}_k H$, respectively. We then have the following classical result of Lazard.

Proposition 5.1 (Thm. 2.4 in [45]). *The canonical projection $G \twoheadrightarrow G/H$ induces a natural isomorphism of graded Lie algebras,*

$$\text{gr}^{\mathcal{F}}(G; \mathbb{k}) / \text{gr}^{\widetilde{\mathcal{F}}}(H; \mathbb{k}) \xrightarrow{\cong} \text{gr}^{\widetilde{\mathcal{F}}}(G/H; \mathbb{k}).$$

5.2. The associated graded Lie algebra. Any group G comes endowed with the lower central series (LCS) filtration $\{\Gamma_k G\}_{k \geq 1}$, defined inductively by $\Gamma_1 G = G$ and

$$(30) \quad \Gamma_{k+1} G = [\Gamma_k G, G].$$

If $\Gamma_k G \neq 1$ but $\Gamma_{k+1} G = 1$, then G is said to be a k -step nilpotent group. In general, though, the LCS filtration does not terminate.

The Lie algebra $\text{gr}(G; \mathbb{k}) = \text{gr}^{\Gamma}(G; \mathbb{k})$ is called the *associated graded Lie algebra* (over \mathbb{k}) of the group G . For instance, if $F = F_n$ is a free group of rank n , then $\text{gr}(F; \mathbb{k})$ is the free graded Lie algebra $\text{lie}(\mathbb{k}^n)$. A group homomorphism $f: G_1 \rightarrow G_2$ induces a morphism of graded Lie algebras, $\text{gr}(f; \mathbb{k}): \text{gr}(G_1; \mathbb{k}) \rightarrow \text{gr}(G_2; \mathbb{k})$; moreover, if f is surjective, then $\text{gr}(f; \mathbb{k})$ is also surjective.

For each $k \geq 2$, the factor group $G/\Gamma_k(G)$ is the maximal $(k-1)$ -step nilpotent quotient of G . The canonical projection $G \rightarrow G/\Gamma_k(G)$ induces an epimorphism $\text{gr}(G; \mathbb{k}) \rightarrow \text{gr}(G/\Gamma_k(G); \mathbb{k})$, which is an isomorphism in degrees $s < k$.

From now on, unless otherwise specified, we will assume that the group G is finitely generated. That is, there is a free group F of finite rank, and an epimorphism $\varphi: F \twoheadrightarrow G$. Let $R = \ker(\varphi)$; then $G = F/R$ is called a presentation for G . Note that the induced morphism $\text{gr}(\varphi; \mathbb{k}): \text{gr}(F; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$ is surjective. Thus, $\text{gr}(G; \mathbb{k})$ is a finitely generated Lie algebra, with generators in degree 1.

Let $H \triangleleft G$ be a normal subgroup, and let $Q = G/H$. If $\bar{\Gamma}_r H = \Gamma_r G \cap H$ is the induced filtration on H , it is readily seen that the filtration $\bar{\Gamma}_r Q = \Gamma_r G / \bar{\Gamma}_r H$ coincides with the LCS filtration on Q . Hence, by Proposition 5.1,

$$(31) \quad \text{gr}(Q; \mathbb{k}) \cong \text{gr}(G; \mathbb{k}) / \text{gr}^{\bar{\Gamma}}(H; \mathbb{k}).$$

Now suppose $G = H \rtimes Q$ is a semi-direct product of groups. In general, there is not much of a relation between the respective associated graded Lie algebras. Nevertheless, we have the following well-known result of Falk and Randell [24], which shows that $\text{gr}(G; \mathbb{k}) = \text{gr}(H; \mathbb{k}) \rtimes \text{gr}(Q; \mathbb{k})$ for ‘almost-direct’ products of groups.

Theorem 5.2 (Thm. 3.1 in [24]). *Let $G = H \rtimes Q$ be a semi-direct product of groups, and suppose Q acts trivially on H_{ab} . Then the filtrations $\{\bar{\Gamma}_r H\}_{r \geq 1}$ and $\{\Gamma_r H\}_{r \geq 1}$ coincide, and there is a split exact sequence of graded Lie algebras,*

$$0 \longrightarrow \text{gr}(H; \mathbb{k}) \longrightarrow \text{gr}(G; \mathbb{k}) \longrightarrow \text{gr}(Q; \mathbb{k}) \longrightarrow 0.$$

5.3. The holonomy Lie algebra. The holonomy Lie algebra of a finitely generated group was introduced by Kohno [41] following the work of K.-T. Chen [15], and further studied in [56, 63, 80].

Definition 5.3. Let G be a finitely generated group. The *holonomy Lie algebra* of G is the holonomy Lie algebra of the cohomology ring $A = H^*(G; \mathbb{k})$, that is,

$$(32) \quad \mathfrak{h}(G; \mathbb{k}) = \text{lie}(H_1(G; \mathbb{k})) / \langle \text{im } \partial_G \rangle,$$

where ∂_G is the dual to the cup-product map $\cup_G: H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \rightarrow H^2(G; \mathbb{k})$.

By construction, $\mathfrak{h}(G; \mathbb{k})$ is a quadratic Lie algebra. If $f: G_1 \rightarrow G_2$ is a group homomorphism, then the induced homomorphism in cohomology, $f^*: H^1(G_2; \mathbb{k}) \rightarrow H^1(G_1; \mathbb{k})$ yields a morphism of graded Lie algebras, $\mathfrak{h}(f; \mathbb{k}): \mathfrak{h}(G_1; \mathbb{k}) \rightarrow \mathfrak{h}(G_2; \mathbb{k})$. Moreover, if f is surjective, then $\mathfrak{h}(f; \mathbb{k})$ is also surjective. Finally, $\mathfrak{h}(G; \mathbb{K}) = \mathfrak{h}(G; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{K}$, for any field extension $\mathbb{k} \subset \mathbb{K}$.

In the definition of the holonomy Lie algebra of G , we used the cohomology ring of a classifying space $K(G, 1)$. As the next lemma shows, we may replace this space by any other connected CW-complex with the same fundamental group.

Lemma 5.4. *Let X be a connected CW-complex with $\pi_1(X) = G$. Then $\mathfrak{h}(H^*(X; \mathbb{k})) \cong \mathfrak{h}(G; \mathbb{k})$.*

Proof. We may construct a classifying space for G by adding cells of dimension 3 and higher to X in a suitable way. The inclusion map, $j: X \rightarrow K(G, 1)$, induces a map on cohomology rings, $j^*: H^*(K(G, 1); \mathbb{k}) \rightarrow H^*(X; \mathbb{k})$, which is an isomorphism in degree 1 and an injection in degree 2. Consequently, j^2 restricts to an isomorphism from $\text{im}(\cup_G)$ to $\text{im}(\cup_X)$. Taking duals, we find that $\text{im}(\partial_X) = \text{im}(\partial_G)$. The conclusion follows. \square

In particular, if K_G is the 2-complex associated to a presentation of G , then $\mathfrak{h}(G; \mathbb{k})$ is isomorphic to $\mathfrak{h}(H^*(K_G; \mathbb{k}))$. Let $\bar{\phi}_n(G) := \dim \mathfrak{h}_n(G; \mathbb{k})$ be the dimensions of the graded pieces of the holonomy Lie algebra of G . The next corollary is an algebraic version of the LCS formula from Papadima and Yuzvinsky [67], but with no formality assumption.

Corollary 5.5. *Let X be a connected CW-complex with $\pi_1(X) = G$, let $A = H^*(X; \mathbb{k})$ be its cohomology algebra, and let \bar{A} be the quadratic closure of A . Then $\prod_{n \geq 1} (1 - t^n)^{\bar{\phi}_n(G)} = \sum_{i \geq 0} b_{ii} t^i$, where $b_{ii} = \dim \text{Ext}_A^i(\mathbb{k}, \mathbb{k})_i$. Moreover, if \bar{A} is a Koszul algebra, then*

$$\prod_{n \geq 1} (1 - t^n)^{\bar{\phi}_n} = \text{Hilb}(\bar{A}, -t).$$

Proof. The first claim follows from Lemma 5.4, the Poincaré–Birkhoff–Witt formula (10), and Löffwall’s formula from Proposition 3.6. The second claim follows from the Koszul duality formula stated in Corollary 3.8. \square

5.4. A comparison map. Once again, let G be a finitely generated group. Although the next lemma is known, we provide a proof in [80].

Lemma 5.6 ([56, 63]). *There exists a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$(33) \quad \Phi_G : \mathfrak{h}(G; \mathbb{k}) \twoheadrightarrow \text{gr}(G; \mathbb{k}),$$

inducing isomorphisms in degrees 1 and 2. Furthermore, this epimorphism is natural with respect to field extensions $\mathbb{k} \subset \mathbb{K}$.

Corollary 5.7. *Let $V = H_1(G; \mathbb{k})$. Suppose the associated graded Lie algebra $\mathfrak{g} = \text{gr}(G; \mathbb{k})$ has presentation $\text{lie}(V)/\mathfrak{r}$. Then the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$ has presentation $\text{lie}(V)/\langle \mathfrak{r}_2 \rangle$, where $\mathfrak{r}_2 = \mathfrak{r} \cap \text{lie}_2(V)$. Furthermore, if $A = U(\mathfrak{g})$, then $\mathfrak{h}(G; \mathbb{k}) = \mathfrak{h}(\bar{A}^1)$.*

Proof. The following natural exact sequence was first noted by Sullivan [82] in a particular case, and proved by Lambe [43] in general,

$$(34) \quad 0 \longrightarrow (\Gamma_2 G / \Gamma_3 G \otimes \mathbb{k})^* \xrightarrow{\beta} H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \xrightarrow{\cup} H^2(G; \mathbb{k}),$$

where β is the dual of Lie bracket product. Taking the dual of this exact sequence, we find that $\text{im}(\partial_G) = \ker(\beta^*)$. Hence, $\langle \mathfrak{r}_2 \rangle = \langle \text{im}(\partial_G) \rangle$ as ideals of $\text{lie}(V)$; thus, $\mathfrak{h}(G; \mathbb{k}) = \text{lie}(V)/\langle \mathfrak{r}_2 \rangle$. The last claim follows from Corollary 3.5. \square

Recall we denote by $\phi_n(G)$ and $\bar{\phi}_n(G)$ the dimensions on the n -th graded pieces of $\text{gr}(G; \mathbb{k})$ and $\mathfrak{h}(G; \mathbb{k})$, respectively. By Lemma 5.6, $\bar{\phi}_n(G) \geq \phi_n(G)$, for all $n \geq 1$, and equality always holds for $n \leq 2$. Nevertheless, these inequalities can be strict for $n \geq 3$.

5.5. Graded-formality. We continue our discussion of the associated graded and holonomy Lie algebras of a finitely generated group with a formality notion that will be important in the sequel.

Definition 5.8. A finitely generated group G is *graded-formal* (over \mathbb{k}) if the canonical projection $\Phi_G : \mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$ is an isomorphism of graded Lie algebras.

This notion was introduced by Lee in [47], where it is called graded 1-formality. Next, we give two alternate definitions, which oftentimes are easier to verify (see [80] for a proof).

Lemma 5.9. *A finitely generated group G is graded-formal over \mathbb{k} if and only if one of the following conditions is satisfied.*

- (1) $\mathrm{gr}(G; \mathbb{k})$ is quadratic.
- (2) $\dim_{\mathbb{k}} \mathfrak{h}_n(G; \mathbb{k}) = \dim_{\mathbb{k}} \mathrm{gr}_n(G; \mathbb{k})$, for all $n \geq 1$.

The lemma implies that the definition of graded formality is independent of the choice of coefficient field \mathbb{k} of characteristic 0. More precisely, we have the following corollary.

Corollary 5.10. *A finitely generated group G is graded-formal over \mathbb{k} if and only if it is graded-formal over \mathbb{Q} .*

Proof. The dimension of a finite-dimensional vector space does not change upon the extensions of scalars $\mathbb{Q} \subset \mathbb{k}$. The conclusion follows at once from Lemma 5.9(2). \square

5.6. Split injections. We are now in a position to state and prove the main result of this section, which proves the first part of Theorem 1.4 from the Introduction.

Theorem 5.11. *Let G be a finitely generated group. Suppose there is a split monomorphism $\iota: K \rightarrow G$. If G is a graded-formal group, then K is also graded-formal.*

Proof. In view of our hypothesis, we have an epimorphism $\sigma: G \twoheadrightarrow K$ such that $\sigma \circ \iota = \mathrm{id}$. In particular, K is also finitely generated. Furthermore, the induced maps $\mathfrak{h}(\iota)$ and $\mathrm{gr}(\iota)$ are also injective.

Let $\pi: F \twoheadrightarrow G$ be a presentation for G . There is then an induced presentation for K , given by the composition $\sigma\pi: F \twoheadrightarrow K$. By Lemma 5.6, there exist epimorphisms Φ_1 and Φ making the following diagram commute:

$$(35) \quad \begin{array}{ccc} \mathfrak{h}(K; \mathbb{k}) & \xrightarrow{\Phi_1} & \mathrm{gr}(K; \mathbb{k}) \\ \downarrow \mathfrak{h}(\iota) & & \downarrow \mathrm{gr}(\iota) \\ \mathfrak{h}(G; \mathbb{k}) & \xrightarrow{\Phi} & \mathrm{gr}(G; \mathbb{k}). \end{array}$$

If the group G is graded-formal, then Φ is an isomorphism of graded Lie algebras. Hence, the epimorphism Φ_1 is also injective, and so K is a graded-formal. \square

Theorem 5.12. *Let $G = K \rtimes Q$ be a semi-direct product of finitely generated groups, and suppose G is graded-formal. Then:*

- (1) *The group Q is graded-formal.*
- (2) *If, moreover, Q acts trivially on K_{ab} , then K is also graded-formal.*

Proof. The first assertion follows at once from Theorem 5.11. So assume Q acts trivially on K_{ab} . By Theorem 5.2, there exists a split exact sequence of graded Lie algebras, which we record in the top row of the next diagram.

$$(36) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{gr}(K; \mathbb{k}) & \xleftarrow{\quad} & \mathrm{gr}(G; \mathbb{k}) & \xleftarrow{\quad} & \mathrm{gr}(Q; \mathbb{k}) \longrightarrow 0 \\ & & \uparrow & & \uparrow \cong & & \uparrow \cong \\ & & \mathfrak{h}(K; \mathbb{k}) & \xleftarrow{\quad} & \mathfrak{h}(G; \mathbb{k}) & \xleftarrow{\quad} & \mathfrak{h}(Q; \mathbb{k}) \longrightarrow 0. \end{array}$$

Let $\iota: K \rightarrow G$ be the inclusion map. By the above, we have an epimorphism, $\sigma: \mathrm{gr}(G; \mathbb{k}) \rightarrow \mathrm{gr}(K; \mathbb{k})$ such that $\sigma \circ \mathrm{gr}(\iota) = \mathrm{id}$. Consequently, $\mathrm{gr}(K; \mathbb{k})$ is finitely generated.

By Corollary 5.7, the map σ induces a morphism $\bar{\sigma}: \mathfrak{h}(G; \mathbb{k}) \rightarrow \mathfrak{h}(G; \mathbb{k})$ such that $\bar{\sigma} \circ \mathfrak{h}(\iota) = \text{id}$. Consequently, $\mathfrak{h}(\iota)$ is injective. Therefore, the morphism $\mathfrak{h}(K; \mathbb{k}) \rightarrow \text{gr}(K; \mathbb{k})$ is also injective. Hence, K is graded-formal. \square

If the hypothesis of Theorem 5.12, part (2) does not hold, the subgroup K may not be graded-formal, even when the group G is 1-formal. We illustrate this phenomenon with an example adapted from [65].

Example 5.13. Let $K = \langle x, y \mid [x, [x, y]], [y, [x, y]] \rangle$ be the discrete Heisenberg group. Consider the semidirect product $G = K \rtimes_{\phi} \mathbb{Z}$, defined by the automorphism $\phi: K \rightarrow K$ given by $x \rightarrow y, y \rightarrow xy$. We have that $b_1(G) = 1$, and so G is 1-formal, yet K is not graded-formal.

5.7. Products and coproducts. We conclude this section with a discussion of the functors gr and \mathfrak{h} and the notion of graded formality behave with respect to products and coproducts.

Lemma 5.14 ([49, 64]). *The functors gr and \mathfrak{h} preserve products and coproducts, that is, we have the following natural isomorphisms of graded Lie algebras,*

$$\begin{cases} \text{gr}(G_1 \times G_2; \mathbb{k}) \cong \text{gr}(G_1; \mathbb{k}) \times \text{gr}(G_2; \mathbb{k}) \\ \text{gr}(G_1 * G_2; \mathbb{k}) \cong \text{gr}(G_1; \mathbb{k}) * \text{gr}(G_2; \mathbb{k}), \end{cases} \quad \text{and} \quad \begin{cases} \mathfrak{h}(G_1 \times G_2; \mathbb{k}) \cong \mathfrak{h}(G_1; \mathbb{k}) \times \mathfrak{h}(G_2; \mathbb{k}) \\ \mathfrak{h}(G_1 * G_2; \mathbb{k}) \cong \mathfrak{h}(G_1; \mathbb{k}) * \mathfrak{h}(G_2; \mathbb{k}). \end{cases}$$

Proof. The first statement on the $\text{gr}(-)$ functor is well-known, while the second statement is the main theorem from [49]. The statements regarding the $\mathfrak{h}(-)$ functor can be found in [64]. \square

Regarding graded-formality, we have the following result, which sharpens and generalizes Lemma 4.5 from Plantiko [69], and proves the first part of Theorem 1.5 from the Introduction.

Proposition 5.15. *Let G_1 and G_2 be two finitely generated groups. Then, the following conditions are equivalent.*

- (1) G_1 and G_2 are graded-formal.
- (2) $G_1 * G_2$ is graded-formal.
- (3) $G_1 \times G_2$ is graded-formal.

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and coproduct $G_1 * G_2$, Theorem 5.11 shows that implications (2) \Rightarrow (1) and (3) \Rightarrow (1) hold. Implications (1) \Rightarrow (2) and (1) \Rightarrow (3) follow from Lemma 5.14 and the naturality of the map Φ from (33). \square

6. MALCEV LIE ALGEBRAS AND FILTERED FORMALITY

In this section we consider the Malcev Lie algebra of a finitely generated group, and study the ensuing notions of filtered formality and 1-formality.

6.1. Prounipotent completions and Malcev Lie algebras. Once again, let G be a finitely generated group, and let $\{\Gamma_k G\}_{k \geq 1}$ be its LCS filtration. The successive quotients of G by these normal subgroups form a tower of finitely generated, nilpotent groups,

$$(37) \quad \cdots \longrightarrow G/\Gamma_4 G \longrightarrow G/\Gamma_3 G \longrightarrow G/\Gamma_2 G = G_{\text{ab}}.$$

Let \mathbb{k} be a field of characteristic 0. It is possible to replace each nilpotent quotient $N_k = G/\Gamma_k G$ by $N_k \otimes \mathbb{k}$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent

group $N_k/\text{tors}(N_k)$ via a procedure which will be discussed in more detail in §10.1. The corresponding inverse limit,

$$(38) \quad \mathfrak{M}(G; \mathbb{k}) = \varprojlim_k ((G/\Gamma_k G) \otimes \mathbb{k}),$$

is a prounipotent, filtered Lie group over \mathbb{k} , which is called the *prounipotent completion*, or *Malcev completion* of G over \mathbb{k} . We denote by $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$ the canonical homomorphism from G to its completion.

Let $\mathfrak{L}ie((G/\Gamma_k G) \otimes \mathbb{k})$ be the Lie algebra of the nilpotent Lie group $(G/\Gamma_k G) \otimes \mathbb{k}$. The pronilpotent Lie algebra

$$(39) \quad \mathfrak{m}(G; \mathbb{k}) := \varprojlim_k \mathfrak{L}ie((G/\Gamma_k G) \otimes \mathbb{k}),$$

with the inverse limit filtration, is called the *Malcev Lie algebra* of G (over \mathbb{k}). By construction, $\mathfrak{m}(-; \mathbb{k})$ is a functor from the category of finitely generated groups to the category of complete, separated, filtered \mathbb{k} -Lie algebras.

6.2. Quillen's construction. A different approach was taken by Quillen in [73, Appendix A]. Let us briefly recall his construction. The group-algebra $\mathbb{k}G$ has a natural Hopf algebra structure, with comultiplication $\Delta: \mathbb{k}G \rightarrow \mathbb{k}G \otimes \mathbb{k}G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, and counit the augmentation map $\varepsilon: \mathbb{k}G \rightarrow \mathbb{k}$. Moreover, the set of *group-like* elements in this Hopf algebra, i.e., those elements x for which $\Delta(x) = x \otimes x$, gets identified with G under the canonical inclusion $G \hookrightarrow \mathbb{k}G$.

The powers of the augmentation ideal $I = \ker \varepsilon$ form a descending, multiplicative filtration of $\mathbb{k}G$ by Hopf ideals. The I -adic completion of the group-algebra, $\widehat{\mathbb{k}G} = \varprojlim_k \mathbb{k}G/I^k$, comes equipped with a descending filtration, whose k -th term is $\widehat{I}^k = \varprojlim_{j \geq k} I^k/I^j$. Identify the completion of $\mathbb{k}G \otimes \mathbb{k}G$ with respect to the natural tensor product filtration with the completed tensor product $\widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}$. Applying the I -adic completion functor to the map Δ yields a comultiplication map $\widehat{\Delta}: \widehat{\mathbb{k}G} \rightarrow \widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}$, which makes $\widehat{\mathbb{k}G}$ into a complete Hopf algebra. It is then apparent that the canonical map to the completion, $\iota: \mathbb{k}G \rightarrow \widehat{\mathbb{k}G}$, is a morphism of filtered Hopf algebras.

An element $x \in \widehat{\mathbb{k}G}$ is called *primitive* if $\widehat{\Delta}x = x \widehat{\otimes} 1 + 1 \widehat{\otimes} x$. The set of all primitive elements in $\widehat{\mathbb{k}G}$, with bracket $[x, y] = xy - yx$, and endowed with the induced filtration, is a Lie algebra, isomorphic to the Malcev Lie algebra of G ,

$$(40) \quad \mathfrak{m}(G; \mathbb{k}) \cong \text{Prim}(\widehat{\mathbb{k}G}).$$

The filtration topology on $\widehat{\mathbb{k}G}$ is a metric topology; hence, the filtration topology on $\mathfrak{m}(G; \mathbb{k})$ is also metrizable, and thus separated. We shall denote by $\text{gr}(\mathfrak{m}(G; \mathbb{k}))$ the associated graded Lie algebra of $\mathfrak{m}(G; \mathbb{k})$ with respect to the induced inverse limit filtration.

The set of all group-like elements in $\widehat{\mathbb{k}G}$, with multiplication inherited from $\widehat{\mathbb{k}G}$, forms a group, denoted $M(G; \mathbb{k})$. This group comes endowed with a complete, separated filtration, with terms $M(G; \mathbb{k}) \cap (1 + \widehat{I}^k)$. As shown by S. Jennings and Quillen, there is a filtration-preserving isomorphism $M(G, \mathbb{k}) \cong \mathfrak{M}(G; \mathbb{k})$, see Massuyeau [58] for details. Furthermore, there is a one-to-one, filtration-preserving correspondence between primitive elements and group-like elements via the exponential

and logarithmic maps

$$(41) \quad \mathfrak{M}(G; \mathbb{k}) \subset 1 + \widehat{I} \begin{array}{c} \xleftarrow{\exp} \\ \xrightarrow{\log} \end{array} \widehat{I} \supset \mathfrak{m}(G; \mathbb{k}) .$$

Restricting the canonical map $\iota: \mathbb{k}G \rightarrow \widehat{\mathbb{k}G}$ to group-like elements, we obtain a homomorphism from G to its pronilpotent completion, $\kappa: G \rightarrow \mathfrak{M}(G; \mathbb{k})$. Composing this homomorphism with the logarithmic map, $\log: \mathfrak{M}(G; \mathbb{k}) \rightarrow \mathfrak{m}(G; \mathbb{k})$, we obtain a filtration-preserving map, $\rho: G \rightarrow \mathfrak{m}(G; \mathbb{k})$. As shown by Quillen in [74], the map ρ induces an isomorphism of graded Lie algebras,

$$(42) \quad \mathrm{gr}(\rho): \mathrm{gr}(G; \mathbb{k}) \xrightarrow{\cong} \mathrm{gr}(\mathfrak{m}(G; \mathbb{k})) .$$

In particular, $\mathrm{gr}(\mathfrak{m}(G; \mathbb{k}))$ is generated in degree 1.

6.3. Minimal models and Malcev Lie algebras. Every group G has a classifying space $K(G, 1)$, which can be chosen to be a connected CW-complex. Such a CW-complex is unique up to homotopy, and thus, up to rational homotopy equivalence. Hence, by the discussion from §4.8 the weak equivalence type of the Sullivan algebra $A = A_{PL}(K(G, 1))$ depends only on the isomorphism type of G . By Theorem 4.5, the DGA $A \otimes_{\mathbb{Q}} \mathbb{k}$ has a 1-minimal model, $\mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{k}, 1)$, unique up to isomorphism. Moreover, the assignment $G \leadsto \mathcal{M}(A \otimes_{\mathbb{Q}} \mathbb{k}, 1)$ is functorial.

Assume now that the group G is finitely generated. Let $\mathcal{M} = \mathcal{M}(G; \mathbb{k})$ be a 1-minimal model of G , with the canonical filtration constructed in (20). The starting point is the finite-dimensional vector space $\mathcal{M}_1^1 = V_1 := H^1(G; \mathbb{k})$. Each sub-DGA \mathcal{M}_i is a Hirsch extension of \mathcal{M}_{i-1} by the finite-dimensional vector space $V_i := \ker(H^2(\mathcal{M}_{i-1}) \rightarrow H^2(A))$.

Define $\mathfrak{L}(G; \mathbb{k}) = \varprojlim_i \mathfrak{L}_i(G; \mathbb{k})$ as the pronilpotent Lie algebra associated to the 1-minimal model $\mathcal{M}(G; \mathbb{k})$ in the manner described in §4.2, and note that the assignment $G \leadsto \mathfrak{L}(G; \mathbb{k})$ is also functorial.

Theorem 6.1 ([12, 31, 83]). *There exist natural isomorphisms of towers of nilpotent Lie algebras,*

$$\begin{array}{ccccc} \cdots & \longleftarrow & \mathfrak{L}_{i-1}(G; \mathbb{k}) & \longleftarrow & \mathfrak{L}_i(G; \mathbb{k}) & \longleftarrow & \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \\ \cdots & \longleftarrow & \mathfrak{m}(G/\Gamma_i G; \mathbb{k}) & \longleftarrow & \mathfrak{m}(G/\Gamma_{i+1} G; \mathbb{k}) & \longleftarrow & \cdots \end{array}$$

Hence, there is a functorial isomorphism $\mathfrak{L}(G; \mathbb{k}) \cong \mathfrak{m}(G; \mathbb{k})$ of complete, filtered Lie algebras.

This functorial isomorphism $\mathfrak{m}(G; \mathbb{k}) \cong \mathfrak{L}(G; \mathbb{k})$, together with the dualization correspondence $\mathfrak{L}(G; \mathbb{k}) \leftrightarrow \mathcal{M}(G; \mathbb{k})$ define adjoint functors between the category of Malcev Lie algebras of finitely generated groups and the category of 1-minimal models of finitely generated groups.

6.4. Filtered formality of groups. We now define the notion of filtered formality for groups (also known as weak formality by Lee [47]), based on the notion of filtered formality for Lie algebras from Definition 2.4.

Definition 6.2. A finitely generated group G is said to be *filtered-formal* (over \mathbb{k}) if its Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is filtered-formal, with respect to the inverse limit filtration.

Here are some more direct ways to think of this notion.

Proposition 6.3. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if one of the following conditions is satisfied.*

- (1) $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$ as filtered Lie algebras.
- (2) $\mathfrak{m}(G; \mathbb{k})$ admits a homogeneous presentation.

Proof. (1) We know from Quillen's isomorphism (42) that $\text{gr}(\mathfrak{m}(G; \mathbb{k})) \cong \text{gr}(G; \mathbb{k})$. The forward implication follows straight from the definitions, while the backward implication follows from Lemma 2.5.

(2) Choose a presentation $\text{gr}(G; \mathbb{k}) = \text{lie}(H_1(G; \mathbb{k}))/\mathfrak{r}$, where \mathfrak{r} is a homogeneous ideal. By Lemma 2.3, we have

$$(43) \quad \mathfrak{m}(G; \mathbb{k}) = \widehat{\text{lie}}(H_1(G; \mathbb{k}))/\bar{\mathfrak{r}},$$

which is a homogeneous presentation for $\mathfrak{m}(G; \mathbb{k})$. Conversely, if (43) holds, then $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{g}}$, where $\mathfrak{g} = \text{lie}(H_1(G; \mathbb{k}))/\mathfrak{r}$. \square

The notion of filtered formality can also be interpreted in terms of minimal models. Let $\mathcal{M}(G; \mathbb{k})$ be the 1-minimal model of G , endowed with the canonical filtration, which is the minimal DGA dual to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ under the correspondence described in §4.2. Likewise, let $\mathcal{N}(G; \mathbb{k})$ be the minimal DGA (generated in degree 1) corresponding to the pronipotent Lie algebra $\widehat{\text{gr}}(G; \mathbb{k})$. Recall that both $\mathcal{M}(G; \mathbb{k})$ and $\mathcal{N}(G; \mathbb{k})$ come equipped with increasing filtrations as in (20), which correspond to the inverse limit filtrations on $\mathfrak{m}(G; \mathbb{k})$ and $\widehat{\text{gr}}(G; \mathbb{k})$, respectively.

Proposition 6.4. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if one of the following conditions is satisfied.*

- (1) *there is a filtration-preserving DGA isomorphism between $\mathcal{M}(G; \mathbb{k})$ and $\mathcal{N}(G; \mathbb{k})$.*
- (2) *there is a DGA isomorphism between $\mathcal{M}(G; \mathbb{k})$ and $\mathcal{N}(G; \mathbb{k})$ inducing the identity on first cohomology.*

Proof. (1) Recall Proposition 6.3 that G is filtered-formal if and only if $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$, as filtered Lie algebras. Dualizing, this condition becomes equivalent to $\mathcal{M}(G; \mathbb{k}) \cong \mathcal{N}(G; \mathbb{k})$, as filtered DGA's.

(2) Recall that G is filtered-formal if and only if $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(\mathfrak{m}(G; \mathbb{k}))$ inducing identity on their associated graded Lie algebras.

Likewise, both \mathcal{M}_1^1 and \mathcal{N}_1^1 can be canonically identified with $\text{gr}_1(G; \mathbb{k})^* = H^1(G; \mathbb{k})$. The desired conclusion follows. \square

Here is another description of filtered formality, suggested to us by R. Porter.

Theorem 6.5. *A finitely generated group G is filtered-formal over \mathbb{k} if and only if the canonical 1-minimal model $\mathcal{M}(G; \mathbb{k})$ is filtered-isomorphic to a 1-minimal model \mathcal{M} with positive Hirsch weights.*

Proof. First suppose G is filtered-formal, and let $\mathcal{N} = \mathcal{N}(G; \mathbb{k})$ be the minimal DGA dual to $\mathfrak{g} = \widehat{\text{gr}}(G; \mathbb{k})$. By Proposition 6.4, this DGA is a 1-minimal model for G . Since by construction $\mathfrak{g} = \widehat{\text{gr}}(\mathfrak{g})$, Lemma 4.6 shows that the differential on \mathcal{N} is homogeneous with respect to the Hirsch weights.

Now suppose \mathcal{M} is a 1-minimal model for G over \mathbb{k} , with homogeneous differential on Hirsch weights. By Lemma 4.6 again, the dual Lie algebra $\mathfrak{g}(\mathcal{M})$ is filtered-formal. On the other hand, the assumption that $\mathcal{M} \cong \mathcal{M}(G; \mathbb{k})$ and Theorem 6.1 together imply that $\mathfrak{g}(\mathcal{M}) \cong \mathfrak{m}(G; \mathbb{k})$. Hence, the group G is filtered-formal by Definition 6.2. \square

We conclude this section by showing that the definition of filtered formality is independent of the choice of coefficient field \mathbb{k} of characteristic 0. We would like to thank Y. Cornulier for asking whether the next result holds, and for pointing out the connection it would have with [16, Thm. 3.14].

Proposition 6.6. *Let G be a finitely generated group, and let $\mathbb{Q} \subset \mathbb{k}$ be a field extension. Then G is filtered-formal over \mathbb{Q} if and only if G is filtered-formal over \mathbb{k} .*

Proof. Write $\mathfrak{m} = \mathfrak{m}(G; \mathbb{Q})$, and let $\mathfrak{g} = \text{gr}(G; \mathbb{Q})$, which we shall identify with $\widehat{\text{gr}}(\mathfrak{m})$. The claim follows from Theorem 2.8. \square

7. FILTERED-FORMALITY AND 1-FORMALITY

In this section, we consider the 1-formality property of finitely generated groups, and the way it relates to Massey products, graded-formality, and filtered-formality. We also study the way various formality properties behave under free and direct products, as well as retracts.

7.1. 1-formality of groups. We start with a basic definition. As usual, let \mathbb{k} be a field of characteristic 0.

Definition 7.1. A finitely generated group G is called *1-formal* (over \mathbb{k}) if a classifying space $K(G, 1)$ is 1-formal over \mathbb{k} .

Since any two classifying spaces for G are homotopy equivalent, the discussion from §4.8 shows that this notion is well-defined. A similar argument shows that the 1-formality property of a path-connected space X depends only on its fundamental group, $G = \pi_1(X)$.

The next, well-known theorem provides an equivalent, purely group-theoretic definition of 1-formality. Although proofs can be found in the literature (see for instance Markl–Papadima [56], Carlson–Toledo [11], and Remark 7.3 below), we provide here an alternative proof, based on Theorem 4.10 and the discussion from §6.3.

Theorem 7.2. *A finitely generated group G is 1-formal over \mathbb{k} if and only if the Malcev Lie algebra of G is isomorphic to the degree completion of the holonomy Lie algebra $\mathfrak{h}(G; \mathbb{k})$.*

Proof. Let $\mathcal{M}(G; \mathbb{k}) = \mathcal{M}(A_{PL}(K(G, 1)), 1) \otimes_{\mathbb{Q}} \mathbb{k}$ be the 1-minimal model of G . The group G is 1-formal if and only if there exists a DGA morphism $\mathcal{M}(G; \mathbb{k}) \rightarrow (H^*(G; \mathbb{k}), d = 0)$ inducing an isomorphism in first cohomology and a monomorphism in second cohomology, i.e., $\mathcal{M}(G; \mathbb{k})$ is a 1-minimal model for $(H^*(G; \mathbb{k}), d = 0)$.

Let $\mathfrak{L}(G; \mathbb{k})$ be the dual Lie algebra of $\mathcal{M}(G; \mathbb{k})$. By Theorem 6.1, the Malcev Lie algebra of G is isomorphic to $\mathfrak{L}(G; \mathbb{k})$. By Theorem 4.10, the degree completion of the holonomy Lie algebra of G is isomorphic to $\mathfrak{L}(G; \mathbb{k})$. This completes the proof. \square

Remark 7.3. Theorem 7.2 admits the following generalization: if G is a finitely generated group, and if (A, d) is a connected DGA with $\dim A^1 < \infty$ whose 1-minimal model is isomorphic to $\mathcal{M}(G; \mathbb{k})$, then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is isomorphic to the completion with respect to the degree filtration of the Lie algebra $\mathfrak{h}(A, d) := \text{lie}((A^1)^*) / \langle \text{im}((d^1)^* + \mu_A^*) \rangle$. A proof of this result is given by Berceanu et al. in [4]; related results can be found in work of Bezrukavnikov [7], Bibby–Hilburn [8], and Polishchuk–Positselski [70].

An equivalent formulation of Theorem 7.2 is given by Papadima and Suciu in [65]: A finitely generated group G is 1-formal over \mathbb{k} if and only if its Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$ is isomorphic to the degree completion of a quadratic Lie algebra, as filtered Lie algebras. For instance, if $b_1(G)$ equals 0 or 1, then G is 1-formal.

Clearly, finitely generated free groups are 1-formal; indeed, if F is such a group, then $\mathfrak{m}(F; \mathbb{k}) \cong \widehat{\text{lie}}(H_1(F; \mathbb{k}))$. Other well-known examples of 1-formal groups include fundamental groups of compact Kähler manifolds, cf. Deligne et al. [18], fundamental groups of complements of complex algebraic hypersurfaces, cf. Kohno [41], and the pure braid groups of surfaces of genus different from 1, cf. Bezrukavnikov [7] and Hain [33].

7.2. Massey products. A well-known obstruction to 1-formality is provided by the higher-order Massey products (introduced in [57]). For our purposes, we will discuss here only triple Massey products of degree 1 cohomology classes.

Let γ_1, γ_2 and γ_3 be cocycles of degrees 1 in the (singular) chain complex $C^*(G; \mathbb{k})$, with cohomology classes $u_i = [\gamma_i]$ satisfying $u_1 \cup u_2 = 0$ and $u_2 \cup u_3 = 0$. That is, we assume there are 1-cochains γ_{12} and γ_{23} such that $d\gamma_{12} = \gamma_1 \cup \gamma_2$ and $d\gamma_{23} = \gamma_2 \cup \gamma_3$. It is readily seen that the 2-cochain $\omega = \gamma_{12} \cup \gamma_3 + \gamma_1 \cup \gamma_{23}$ is, in fact, a cocycle. The set of all cohomology classes $[\omega]$ obtained in this way is the *Massey triple product* $\langle u_1, u_2, u_3 \rangle$ of the classes u_1, u_2 and u_3 . Due to the ambiguity in the choice of γ_{12} and γ_{23} , the Massey triple product $\langle u_1, u_2, u_3 \rangle$ is a representative of the coset

$$(44) \quad H^2(G; \mathbb{k}) / (u_1 \cup H^1(G; \mathbb{k}) + H^1(G; \mathbb{k}) \cup u_3).$$

Remark 7.4. In [71, Thm. 2], Porter gave a topological method for computing cup products and higher-order Massey products in $H^2(G; \mathbb{k})$. Building on work of Dwyer [21], Fenn and Sjerve [27] gave explicit formulas for Massey products in a commutator-relator group. For instance, suppose $G = \langle \mathbf{x} \mid r \rangle$, where the single relator r belongs to $[F, F]$ and is not a proper power. Let $I = (i_1, \dots, i_k)$, and suppose $\epsilon_{i_s, \dots, i_{t-1}}(r) = 0$ for all $1 \leq s < t \leq k+1$ with $(s, t) \neq (1, k+1)$, where $\epsilon_{i_s, \dots, i_{t-1}}$ is the composition of the augmentation map with the iterated Fox derivative $\partial_{i_s, \dots, i_{t-1}}$. Then the evaluation of the Massey product $\langle -u_{i_1}, \dots, -u_{i_k} \rangle$ on the homology class $[r] \in H_2(G; \mathbb{Z})$ equals $\epsilon_I(r)$.

If a group G is 1-formal, then all triple Massey products vanish in the quotient \mathbb{k} -vector space from (44). However, if G is only graded-formal, these Massey products need not vanish. As we shall see in Example 7.5, even a one-relator group G may be graded-formal, yet not 1-formal.

Example 7.5. Let $G = \langle x_1, \dots, x_5 \mid [x_1, x_2][x_3, [x_4, x_5]] = 1 \rangle$. From [80], the group G is graded-formal. On the other hand, Remark 7.4 shows that G admits a non-trivial triple Massey product of the form $\langle u_3, u_4, u_5 \rangle$. Thus, G is not 1-formal, and so G is not filtered-formal.

7.3. Filtered formality, graded formality and 1-formality. The next result pulls together the various formality notions for groups, and establishes the basic relationship among them.

Proposition 7.6. *A finitely generated group G is 1-formal if and only if G is graded-formal and filtered-formal.*

Proof. First suppose G is 1-formal. Then, by Theorem 7.2, $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{h}}(G; \mathbb{k})$, and thus, $\text{gr}(G; \mathbb{k}) \cong \mathfrak{h}(G; \mathbb{k})$, by (42). Hence, G is graded-formal, by Lemma 5.9(1). It follows that $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$, and hence G is filtered-formal, by Proposition 6.3.

Now suppose G filtered-formal. Then, by Proposition 6.3, we have that $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}}(G; \mathbb{k})$. Thus, if G is also graded-formal, $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathfrak{h}}(G; \mathbb{k})$. Hence, G is 1-formal. \square

Using this proposition, together with Proposition 6.6 and Corollary 5.10, we obtain the following corollary.

Corollary 7.7. *A finitely generated group G is 1-formal over \mathbb{Q} if and only if G is 1-formal over \mathbb{k} .*

In other words, the 1-formality property of a finitely generated group is independent of the choice of coefficient field of characteristic 0.

In general, a filtered-formal group need not be 1-formal. Examples include some of the free nilpotent groups from Example 10.1 or the unipotent groups from Example 10.8. In fact, the triple Massey products in the cohomology of a filtered-formal group need not vanish (modulo indeterminacy).

Example 7.8. Let $G = F_2/\Gamma_3 F_2 = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = [x_2, [x_1, x_2]] = 1 \rangle$ be the Heisenberg group. Then G is filtered-formal, yet has non-trivial triple Massey products $\langle u_1, u_1, u_2 \rangle$ and $\langle u_2, u_1, u_2 \rangle$ in $H^2(G; \mathbb{k})$. Hence, G is not graded-formal.

As shown by Hain in [33, 34] the Torelli groups in genus 4 or higher are 1-formal, but the Torelli group in genus 3 is filtered-formal, yet not graded-formal. The next two examples show that there are groups which are graded-formal but not filtered-formal.

Example 7.9. In [2], Bartholdi et al. consider two infinite families of groups. The first are the quasitriangular groups QTr_n , which have presentations with generators x_{ij} ($1 \leq i \neq j \leq n$), and relations $x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}$ and $x_{ij}x_{kl} = x_{kl}x_{ij}$ for distinct i, j, k, l . The second are the triangular groups Tr_n , each of which is the quotient of QTr_n by the relations of the form $x_{ij} = x_{ji}$ for $i \neq j$. As shown by Lee in [47], the groups QTr_n and Tr_n are all graded-formal. On the other hand, as indicated in [2], these groups are non-1-formal (and hence, not filtered-formal) for all $n \geq 4$. A detailed proof of this fact will be given in [79].

Example 7.10. Let G be the group with generators x_1, \dots, x_4 and relators $[x_2, x_3]$, $[x_1, x_4]$, and $[x_1, x_3][x_2, x_4]$. From [80], G is graded-formal. On the other hand, using the Tangent Cone theorem of Dimca et al. [19], one can show that the group G is not 1-formal. Therefore, G is not filtered-formal.

7.4. Examples from link theory. Let $L = (L_1, \dots, L_n)$ be an n -component link in S^3 . The link group, $G = \pi_1(X)$, is the fundamental group of the complement, $X = S^3 \setminus \bigcup_{i=1}^n L_i$. In general, a link group (even a pure braid link group) is not 1-formal. This phenomenon was first detected by W.S. Massey by means of his higher-order products [57], but graded and especially filtered formality can be even harder to detect. The graded Lie algebras in the following two examples were first given by Hain [32]. In [80], these Lie algebras were also computed based on the work of Anick [1]. This implies that the groups in the following examples are not graded-formal, hence not 1-formal. Furthermore, Hain's result also implies that the following two groups are filtered-formal.

Example 7.11. Let L be the Borromean rings. This is the 3-component link obtained by closing up the pure braid $[A_{1,2}, A_{2,3}] \in P'_3$, where $A_{i,j}$ denote the standard generators of the pure braid group. Alternatively, the non-1-formality of G can be detected by the triple Massey products $\langle u, v, w \rangle$ and $\langle w, v, u \rangle$.

Example 7.12. Let L be the Whitehead link. This is a 2-component link with linking number 0. Its link group is the 1-relator group $G = \langle x, y \mid r \rangle$, where

$$r = x^{-1}y^{-1}xyx^{-1}yxy^{-1}xyx^{-1}y^{-1}xy^{-1}x^{-1}y.$$

The non-1-formality of G can also be detected by suitable fourth-order Massey products.

Next, we give an example of a link group which is graded-formal, yet not filtered-formal.

Example 7.13. Let L be the link of great circles in S^3 corresponding to the arrangement of transverse planes through the origin of \mathbb{R}^4 denoted as $\mathcal{A}(31425)$ in Matei–Suciu [59]. Then L is a pure braid link of 5 components, with linking graph the complete graph K_5 ; moreover, the link group G is isomorphic to the semidirect product $F_4 \rtimes_{\alpha} F_1$, where $\alpha = A_{1,3}A_{2,3}A_{2,4} \in P_4$. From [80], based on the work of Berceanu–Papadima [5], the group G is graded-formal. On the other hand, as noted by Dimca et al. in [19, Example 8.2], the Tangent Cone theorem does not hold for this group, and thus G is not 1-formal. Consequently, G is not filtered-formal.

7.5. Propagation of filtered formality. The next theorem shows that filtered formality is inherited upon taking nilpotent quotients. In §10, we will focus on exploring the formality properties of torsion-free nilpotent groups.

Theorem 7.14. *Let G be a finitely generated group, and suppose G is filtered-formal. Then all the nilpotent quotients $G/\Gamma_i(G)$ are filtered-formal.*

Proof. Set $\mathfrak{g} = \text{gr}(G; \mathbb{k})$ and $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$, and write $\mathfrak{g} = \bigoplus_{k \geq 1} \mathfrak{g}_k$. Then, for each $i \geq 1$, the canonical projection $\phi_i: G \rightarrow G/\Gamma_i G$ induces an epimorphism of complete, filtered Lie algebras, $\mathfrak{m}(\phi_i): \mathfrak{m} \twoheadrightarrow \mathfrak{m}(G/\Gamma_i G; \mathbb{k})$. In each degree $k \geq i$, we have that $\widehat{\Gamma}_k \mathfrak{m}(G/\Gamma_i G; \mathbb{k}) = 0$, and so $\mathfrak{m}(\phi_i)(\widehat{\Gamma}_k \mathfrak{m}) = 0$. Therefore, there exists an induced epimorphism

$$(45) \quad \Phi_{k,i}: \mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m} \twoheadrightarrow \mathfrak{m}(G/\Gamma_i G; \mathbb{k}).$$

Passing to the associated graded, we obtain an epimorphism $\text{gr}(\mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m}) \twoheadrightarrow \text{gr}(\mathfrak{m}(G/\Gamma_i G; \mathbb{k}))$, which is readily seen to be an isomorphism for $k = i$. Using now Lemma 2.2, we conclude that the map $\Phi_{i,i}$ is an isomorphism of (complete, separated) filtered Lie algebras.

On the other hand, our filtered-formality assumption on G allows us to identify $\mathfrak{m} \cong \widehat{\mathfrak{g}} = \prod_{k \geq 1} \mathfrak{g}_k$. Let $\iota: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ be the canonical morphism. By construction, we have isomorphisms $\iota_k: \mathfrak{g}/\Gamma_k \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}/\widehat{\Gamma}_k \widehat{\mathfrak{g}}$ for all $k \geq 1$. Thus, $\mathfrak{m}/\widehat{\Gamma}_k \mathfrak{m} \cong \widehat{\mathfrak{g}}/\widehat{\Gamma}_k \widehat{\mathfrak{g}} \cong \mathfrak{g}/\Gamma_k \mathfrak{g}$, for all $k \geq 1$. Using these identifications for $k = i$, together with the isomorphism $\Phi_{i,i}$ from above, we obtain isomorphisms

$$(46) \quad \mathfrak{m}(G/\Gamma_i G; \mathbb{k}) \cong \mathfrak{g}/\Gamma_i \mathfrak{g} \cong \text{gr}(G/\Gamma_i G; \mathbb{k}).$$

This shows that the nilpotent quotient $G/\Gamma_i G$ is filtered-formal, and we are done. \square

Proposition 7.15. *Suppose $\phi: G_1 \rightarrow G_2$ is a homomorphism between two finitely generated groups, inducing an isomorphism $H_1(G_1; \mathbb{k}) \rightarrow H_1(G_2; \mathbb{k})$ and an epimorphism $H_2(G_1; \mathbb{k}) \rightarrow H_2(G_2; \mathbb{k})$. Then we have the following statements.*

- (1) *If G_2 is 1-formal, then G_1 is also 1-formal.*
- (2) *If G_2 is filtered-formal, then G_1 is also filtered-formal.*
- (3) *If G_2 is graded-formal, then G_1 is also graded-formal.*

Proof. A celebrated theorem of Stallings [77] (see also Dwyer [21] and Freedman et al. [29]) insures that the homomorphism ϕ induces isomorphisms $\phi_k: (G_1/\Gamma_k G_1) \otimes \mathbb{k} \rightarrow (G_2/\Gamma_k G_2) \otimes \mathbb{k}$, for all $k \geq 1$. Hence, ϕ induces an isomorphism $\mathfrak{m}(\phi): \mathfrak{m}(G_1; \mathbb{k}) \rightarrow \mathfrak{m}(G_2; \mathbb{k})$ between the respective Malcev completions, thereby proving claim (1). Using now the isomorphism (42), the other two claims follow at once. \square

7.6. Split injections. We are now ready to state and prove the main result of this section, which completes the proof of Theorem 1.4 from the Introduction.

Theorem 7.16. *Let G be a finitely generated group. Suppose there is a split monomorphism $\iota: K \rightarrow G$. The following statements then hold.*

- (1) *If G is filtered-formal, then K is also filtered-formal.*
- (2) *If G is 1-formal, then K is also 1-formal.*

Proof. By hypothesis, we have an epimorphism $\sigma: G \twoheadrightarrow K$ such that $\sigma \circ \iota = \text{id}$. It follows that the induced maps $\mathfrak{m}(\iota)$ and $\widehat{\text{gr}}(\iota)$ are also split injections.

Let $\pi: F \twoheadrightarrow G$ be a presentation for G . We then have an induced presentation for K , given by the composition $\pi_1 := \sigma\pi: F \twoheadrightarrow K$. There is also a map $\iota_1: F \rightarrow F$ which is a lift of ι , that is, $\iota\pi_1 = \pi\iota_1$. Consider the following diagram (for simplicity, we will suppress the zero-characteristic coefficient field \mathbb{k} from the notation).

$$(47) \quad \begin{array}{ccccccc} & & J_1 & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \widehat{\text{gr}}(K) \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \widehat{\text{gr}}(\iota) \\ I_1 & \hookrightarrow & \widehat{\text{lie}}(F) & \xrightarrow{\text{id}} & \widehat{\text{lie}}(F) & \twoheadrightarrow & \mathfrak{m}(K) \\ & \searrow & \downarrow & & \downarrow & & \downarrow \Phi_1 \\ & & J & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \widehat{\text{gr}}(G) \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \widehat{\text{gr}}(\iota) \\ I & \hookrightarrow & \widehat{\text{lie}}(F) & \xrightarrow{\text{id}} & \widehat{\text{lie}}(F) & \twoheadrightarrow & \mathfrak{m}(G) \\ & \searrow & \downarrow & & \downarrow & & \downarrow \Phi \\ & & I & \hookrightarrow & \widehat{\text{lie}}(F) & \twoheadrightarrow & \mathfrak{m}(G) \end{array}$$

We have $\mathfrak{m}(\iota_1) = \widehat{\text{gr}}(\iota_1)$. By assumption, G is filtered-formal; hence, there exists a filtered Lie algebra isomorphism $\Phi: \mathfrak{m}(G) \rightarrow \widehat{\text{gr}}(G)$ as in diagram (47), which induces the identity on associated graded algebras. It follows that Φ is induced from the identity map of $\widehat{\text{lie}}(F)$ upon projecting onto source and target, i.e., the bottom right square in the diagram commutes.

First, we show that the identity map $\text{id}: \widehat{\text{lie}}(F) \rightarrow \widehat{\text{lie}}(F)$ in the above diagram induces an inclusion map $I_1 \rightarrow J_1$. Suppose there is an element $c \in \widehat{\text{lie}}(F)$ such that $c \in I_1$ and $c \notin J_1$, i.e., $[c] = 0$ in $\mathfrak{m}(K)$ and $[c] \neq 0$ in $\widehat{\text{gr}}(G)$. Since $\widehat{\text{gr}}(\iota)$ is injective, we have that $\widehat{\text{gr}}(\iota)([c]) \neq 0$, i.e., $\widehat{\text{gr}}(\iota_1)(c) \notin I$. We also have $\mathfrak{m}(\iota)([c]) = 0 \in \mathfrak{m}(G)$, i.e., $\mathfrak{m}(\iota_1)(c) \in J$. This contradicts the fact that the inclusion $I \hookrightarrow J$ is induced by the identity map. Thus, $I_1 \subset J_1$.

In view of the above, we may define a Lie algebra morphism $\Phi_1: \mathfrak{m}(K) \rightarrow \widehat{\text{gr}}(K)$ as the quotient of the identity on $\widehat{\text{lie}}(F)$. By construction, Φ_1 is an epimorphism. We also have $\widehat{\text{gr}}(\iota) \circ \Phi_1 = \Phi \circ \mathfrak{m}(\iota)$. Since the maps $\mathfrak{m}(\iota)$, $\widehat{\text{gr}}(\iota)$ and Φ are all injective, the map Φ_1 is also injective. Therefore, Φ_1 is an isomorphism, and so the group K is filtered-formal.

Finally, part (2) follows at once from part (1) and Theorem 5.11. \square

This completes the proof of Theorem 1.4 from the Introduction. As we shall see in Example 7.9, this theorem is useful for deciding whether certain infinite families of groups are 1-formal.

We now proceed with the proof of Theorem 1.5. First, we need a lemma.

Lemma 7.17 ([19]). *Let G_1 and G_2 be two finitely generated groups. Then $\mathfrak{m}(G_1 \times G_2; \mathbb{k}) \cong \mathfrak{m}(G_1; \mathbb{k}) \times \mathfrak{m}(G_2; \mathbb{k})$ and $\mathfrak{m}(G_1 * G_2; \mathbb{k}) \cong \mathfrak{m}(G_1; \mathbb{k}) \hat{*} \mathfrak{m}(G_2; \mathbb{k})$.*

Proposition 7.18. *For any two finitely generated groups G_1 and G_2 , the following conditions are equivalent.*

- (1) G_1 and G_2 are filtered-formal.
- (2) $G_1 * G_2$ is filtered-formal.
- (3) $G_1 \times G_2$ is filtered-formal.

Proof. Since there exist split injections from G_1 and G_2 to the product $G_1 \times G_2$ and coproduct $G_1 * G_2$, we may apply Theorem 7.16 to conclude that implications (2) \Rightarrow (1) and (3) \Rightarrow (1) hold. Implications (1) \Rightarrow (2) and (1) \Rightarrow (3) follow from Lemmas 2.9, 2.10, and 7.17. \square

Remark 7.19. As we shall see in Example 7.13, the implication (1) \Rightarrow (3) from Proposition 7.18 cannot be strengthened from direct products to arbitrary semi-direct products. More precisely, there exist split extensions of the form $G = F_n \rtimes_{\alpha} \mathbb{Z}$, for certain automorphisms $\alpha \in \text{Aut}(F_n)$, such that the group G is not filtered-formal, although of course both F_n and \mathbb{Z} are 1-formal.

Corollary 7.20. *Suppose G_1 and G_2 are finitely generated groups such that G_1 is not graded-formal and G_2 is not filtered-formal. Then the product $G_1 \times G_2$ and the free product $G_1 * G_2$ are neither graded-formal, nor filtered-formal.*

Proof. Follows at once from Propositions 5.15 and 7.18. \square

As mentioned in the Introduction, concrete examples of groups which do not possess either formality property can be obtained by taking direct products of groups which enjoy one property but not the other.

8. EXPANSIONS OF GROUPS

In this section, we relate the 1-formality and filtered formality properties of a group to expansions of its group algebra.

8.1. Expansions and complete Hopf algebras. Using K.T. Chen's theory of formal power series connections and their induced monodromy representations, X.-S. Lin studied in [50] expansions of fundamental groups of smooth manifolds. Recently, D. Bar-Natan [3] generalized the idea of expansion and explored the Taylor expansion of an arbitrary ring. For our purposes here, we recall the definitions from [3] in the case when the ring in question is the group algebra $\mathbb{k}G$ of a finitely generated group G , over a field \mathbb{k} of characteristic 0.

Given a map $f: G \rightarrow R$, where R is a ring, we will denote by $\tilde{f}: \mathbb{k}G \rightarrow R$ its linear extension to the group algebra. Let $\text{gr}(\mathbb{k}G) = \bigoplus_{k \geq 0} I^k / I^{k+1}$ be the associated graded algebra of $\mathbb{k}G$ with respect to the augmentation ideal $I = \ker(\epsilon: \mathbb{k}G \rightarrow \mathbb{k})$, and let $\widehat{\text{gr}}(\mathbb{k}G) = \prod_{k \geq 0} I^k / I^{k+1}$ be its degree completion. The algebra $\text{gr}(\mathbb{k}G)$ comes endowed with the degree filtration, $\mathcal{F}_k(\text{gr}(\mathbb{k}G)) = \bigoplus_{j \geq k} I^j / I^{j+1}$, while $\widehat{\text{gr}}(\mathbb{k}G)$ comes endowed with the inverse limit filtration, $\widehat{\mathcal{F}}_k(\widehat{\text{gr}}(\mathbb{k}G)) = \prod_{j \geq k} I^j / I^{j+1}$. Clearly, the associated graded algebra of $\widehat{\text{gr}}(\mathbb{k}G)$ is canonically identified $\text{gr}(\mathbb{k}G)$.

Definition 8.1. A (multiplicative) *expansion* of a group G is a map $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ such that the linear extension $\bar{E}: \mathbb{k}G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ is a filtration-preserving algebra morphism with the property that $\text{gr}(\bar{E}) = \text{id}$. Furthermore, we say that the expansion E is *faithful* if E is injective.

Alternatively, an expansion of G is a (multiplicative) monoid map $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ such that the following property holds: If $f \in I^k \setminus I^{k+1}$, then $\bar{E}(f)$ begins with $[f] \in I^k / I^{k+1}$, that is, $\bar{E}(f) = (0, \dots, 0, [f], *, *, \dots)$.

8.2. Taylor expansions and complete Hopf algebras. Recall now from §6.2 the work of Quillen [73], who showed that $\mathbb{k}G$ is a Hopf algebra, with comultiplication map $\Delta: \mathbb{k}G \rightarrow \mathbb{k}G \otimes \mathbb{k}G$ given by $\Delta(g) = g \otimes g$ for $g \in G$, while the I -adic completion $\widehat{\mathbb{k}G}$ is a complete Hopf algebra, with comultiplication map $\widehat{\Delta}: \widehat{\mathbb{k}G} \rightarrow \widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G}$. The powers of the augmentation ideal I give a filtration of $\mathbb{k}G$, while $\widehat{\mathbb{k}G}$ is equipped with the filtration \widehat{I}^k . Identifying now the associated algebra $\text{gr}(\mathbb{k}G \otimes \mathbb{k}G)$ with $\text{gr}(\mathbb{k}G) \otimes \text{gr}(\mathbb{k}G)$, we see that the degree completion $\widehat{\text{gr}}(\mathbb{k}G)$ is also a complete Hopf algebra, with comultiplication map $\widehat{\Delta} := \widehat{\text{gr}}(\Delta): \widehat{\text{gr}}(\mathbb{k}G) \rightarrow \widehat{\text{gr}}(\mathbb{k}G) \widehat{\otimes} \widehat{\text{gr}}(\mathbb{k}G)$.

As in [3], an expansion E of G is called a *Taylor expansion* (or, a *group-like expansion*) if it sends all elements of G to group-like elements of $\widehat{\text{gr}}(\mathbb{k}G)$, that is,

$$(48) \quad \bar{\Delta}(E(g)) = E(g) \widehat{\otimes} E(g)$$

for all $g \in G$. Equivalently, the expansion E is *co-multiplicative*, i.e., the following diagram commutes:

$$(49) \quad \begin{array}{ccc} \mathbb{k}G & \xrightarrow{\Delta} & \mathbb{k}G \otimes \mathbb{k}G \\ \downarrow \bar{E} & & \downarrow \bar{E} \otimes \bar{E} \\ \widehat{\text{gr}}(\mathbb{k}G) & \xrightarrow{\widehat{\Delta}} & \widehat{\text{gr}}(\mathbb{k}G) \widehat{\otimes} \widehat{\text{gr}}(\mathbb{k}G). \end{array}$$

Lemma 8.2. *A Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ induces a filtration-preserving isomorphism of complete Hopf algebras, $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, such that $\text{gr}(\widehat{E})$ is the identity.*

Proof. As in the above definition, the expansion E induces a filtration-preserving algebra morphism, $\bar{E}: \mathbb{k}G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. Applying the I -adic completion functor, we obtain an algebra morphism, $\widehat{E}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$. By the above discussion, the expansion E is group-like if and only if \bar{E} is co-multiplicative. Applying the completion functor to diagram (49) yields another commuting diagram,

$$(50) \quad \begin{array}{ccc} \widehat{\mathbb{k}G} & \xrightarrow{\widehat{\Delta}} & \widehat{\mathbb{k}G} \widehat{\otimes} \widehat{\mathbb{k}G} \\ \downarrow \widehat{E} & & \downarrow \widehat{E} \widehat{\otimes} \widehat{E} \\ \widehat{\text{gr}}(\mathbb{k}G) & \xrightarrow{\widehat{\Delta}} & \widehat{\text{gr}}(\mathbb{k}G) \widehat{\otimes} \widehat{\text{gr}}(\mathbb{k}G). \end{array}$$

Since \bar{E} is filtration-preserving and $\text{gr}(\bar{E}) = \text{id}$, this implies that the Hopf algebra morphism \widehat{E} preserves filtrations and that $\text{gr}(\widehat{E}) = \text{id}$. An argument as in the proof of Lemma 2.2 now shows that \widehat{E} is an isomorphism. \square

Lemma 8.3. *A filtration-preserving isomorphism of complete Hopf algebras, $\phi: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, induces a Taylor expansion $E: G \rightarrow \widehat{\text{gr}}(G)$.*

Proof. As in the proof of Lemma 2.5, we see that the isomorphism ϕ induces a filtration-preserving isomorphism of complete Hopf algebras, $\tilde{\phi}: \widehat{\mathbb{k}G} \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$, such that $\text{gr}(\tilde{\phi}) = \text{id}$. Let $E: G \rightarrow \widehat{\text{gr}}(\mathbb{k}G)$ be the composite

$$(51) \quad G \hookrightarrow \mathbb{k}G \xrightarrow{\iota} \widehat{\mathbb{k}G} \xrightarrow{\tilde{\phi}} \widehat{\text{gr}}(\mathbb{k}G).$$

Since both $\tilde{\phi}$ and ι are Hopf algebra maps, and the inclusion $G \hookrightarrow \mathbb{k}G$ is a monoid map sending G to the group-like elements of $\mathbb{k}G$, the composite E is also a monoid map. It is clear that $\widehat{E} = \tilde{\phi}$

and $\bar{E} = \tilde{\phi} \circ \iota$. Since both $\tilde{\phi}$ and ι are filtration-preserving, and $\text{gr}(\iota) = \text{gr}(\bar{E}) = \text{id}$, we infer that \bar{E} is filtration-preserving, and $\text{gr}(\bar{E}) = \text{id}$. Finally, by construction, E is a group-like expansion. \square

Lemmas 8.2 and 8.3 generalize Massuyeau's [58, Prop. 2.10], from a finitely generated free group to an arbitrary finitely generated group. These two lemmas have the following immediate corollary.

Corollary 8.4. *A finitely generated group G has a Taylor expansion if and only if there is an isomorphism of filtered Hopf algebras, $\widehat{\mathbb{k}G} \cong \widehat{\text{gr}(\mathbb{k}G)}$.*

8.3. Taylor expansions and formality properties. We now relate the preceding concepts to the filtered formality and 1-formality properties of groups.

Theorem 8.5. *A finitely generated group G has a Taylor expansion if and only if G is filtered-formal.*

Proof. First suppose G has a Taylor expansion. Then, by Lemma 8.2, there is a filtration-preserving Hopf algebra isomorphism between $\widehat{\mathbb{k}G}$ and $\widehat{\text{gr}(\mathbb{k}G)}$, inducing the identity on $\text{gr}(\mathbb{k}G)$. Now recall that $\widehat{\mathbb{k}G} \cong U(\mathfrak{m}(G; \mathbb{k}))$ and $\widehat{\text{gr}(\mathbb{k}G)} \cong U(\widehat{\text{gr}(G; \mathbb{k})})$, as filtered Hopf algebras. Taking primitives, we obtain a filtration-preserving isomorphism of complete Lie algebras, $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\text{gr}(G; \mathbb{k})}$, inducing the identity on $\text{gr}(G; \mathbb{k})$. Hence, G is filtered-formal.

Now suppose that G is filtered-formal. There is then an isomorphism of filtered, complete Lie algebras, $\alpha: \mathfrak{m}(G; \mathbb{k}) \rightarrow \widehat{\text{gr}(G; \mathbb{k})}$, such that $\text{gr}(\alpha) = \text{id}$. Taking universal enveloping algebras, we obtain an isomorphism of filtered, complete Hopf algebras $\phi: \widehat{\mathbb{k}G} \cong \widehat{\text{gr}(\mathbb{k}G)}$, such that $\text{gr}(\phi) = \text{id}$. It follows from Lemma 8.3 that G has a Taylor expansion. \square

Next, we obtain an equivalent description for 1-formality of groups.

Corollary 8.6. *A finitely generated group G is 1-formal if and only if there is a Taylor expansion $G \rightarrow \widehat{\text{gr}(\mathbb{k}G)}$ and $\text{gr}(\mathbb{k}G)$ is a quadratic algebra.*

Proof. We know from Proposition 7.6 that G is 1-formal if and only if G is filtered-formal and graded-formal. By Theorem 8.5, G is filtered-formal if and only if it has a Taylor expansion. On the other hand, it follows from Lemma 5.9(1) that G is graded-formal if and only if $\text{gr}(G; \mathbb{k}) \cong \mathfrak{h}(G; \mathbb{k})$. As shown in [47, §2.2.3], this latter condition is equivalent to the quadraticity of $\text{gr}(\mathbb{k}G)$. This completes the proof. \square

Corollary 8.7. *The existence of a Taylor expansion is preserved under field extensions, finite products and coproducts, split injections, nilpotent quotients and solvable quotients of groups.*

Proof. In view of Theorem 8.5, our claims follow from Propositions 6.6 (for field extensions) and 7.18 (for products and coproducts), as well as Theorems 7.16 (for split injections), 7.14 (for nilpotent quotients), and 9.3 (for solvable quotients). \square

A group G is said to be *residually torsion-free nilpotent* (for short RTFN) if for any $g \in G$, $g \neq 1$, there exists a torsion-free nilpotent group Q , and an epimorphism $\psi: G \rightarrow Q$ such that $\psi(g) \neq 1$. Equivalently, G is residually torsion-free nilpotent if and only if $\bigcap_{k \geq 1} \tau_k G = \{1\}$, where

$$(52) \quad \tau_k G = \{g \in G \mid g^n \in \Gamma_k G, \text{ for some } n \in \mathbb{N}\}.$$

By [68, Ch. VI, Thm. 2.26], a group G is residually torsion-free nilpotent if and only if the group-algebra $\mathbb{k}G$ is residually nilpotent, that is, $\bigcap_{k \geq 1} I^k = \{0\}$. Therefore, if G is finitely generated, the RTFN condition is equivalent to the injectivity of the canonical map to the pronilpotent completion, $\kappa: G \rightarrow \mathfrak{M}(G, \mathbb{k})$.

Corollary 8.8. *A finitely generated group G has a faithful Taylor expansion if and only if G is residually torsion-free nilpotent and filtered-formal.*

As shown by Magnus, all free groups are residually torsion-free nilpotent. Furthermore, as shown by Hain [33] and Berceanu–Papadima [6], the Torelli group $\mathrm{IA}_n = \ker(\mathrm{Aut}(F) \rightarrow \mathrm{Aut}(F_{\mathrm{ab}}))$ is residually torsion-free nilpotent, for all $n \geq 1$. Hence, all its subgroups, such as the pure braid group P_n , the McCool group wP_n , and the upper McCool group wP_n^+ also enjoy this property. We refer to [78] for more details on this subject.

8.4. Discussion and examples. Let F be a finitely-generated free group on generators x_1, \dots, x_n . The complete Hopf algebra $\widehat{\mathrm{gr}}(\mathbb{k}F)$ can be identified with $\mathbb{k}\langle\langle X_1, \dots, X_n \rangle\rangle$, the power series ring over \mathbb{k} in n non-commuting variables. There is a well-known faithful expansion, $M: F \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}F)$, called the Magnus expansion, which is given by $M(x_i) = 1 + X_i$, see [54]. However, the Magnus expansion is not co-multiplicative if $n > 1$; thus, it is not a Taylor expansion. On the other hand, the power series expansion, defined by $x_i \mapsto \exp(X_i)$, is a Taylor expansion, see [50].

Assume now that G is a group which admits a finite presentation of the form $G = F/R$. Using a Taylor expansion for the finitely generated free group F , we may find a presentation for the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$, using the approach of Papadima [62] and Massuyeau [58].

Theorem 8.9 ([58, 62]). *Let G be a group with generators x_1, \dots, x_n and relators r_1, \dots, r_m . Let E be a Taylor expansion of the free group $F = \langle x_1, \dots, x_n \rangle$. There exists then a unique filtered Lie algebra isomorphism $\mathfrak{m}(G; \mathbb{k}) \cong \widehat{\mathrm{lie}}(F_{\mathbb{k}})/\langle\langle W \rangle\rangle$, where $\langle\langle W \rangle\rangle$ is the closed ideal of $\widehat{\mathrm{lie}}(F_{\mathbb{k}})$ generated by the subset $\{\log(E(r_1)), \dots, \log(E(r_m))\} \subset \mathrm{lie}(F_{\mathbb{k}})$.*

We conclude this section with two families of groups which admit explicit Taylor expansions.

Example 8.10. The reduced free group RF_n , introduced by J. Milnor in his study of link homotopy, is the quotient of the free group $F_n = \langle x_1, \dots, x_n \rangle$ by the normal subgroup generated by all elements of the form $[x_i, g x_i g^{-1}]$. In [50], Lin showed that RF_n has a Taylor expansion induced by the power series expansion of F_n . It follows from Theorem 8.5 that the group RF_n is filtered-formal.

Example 8.11. Let Π_g be the fundamental group of the closed, orientable surface of genus $g \geq 1$. It is well-known that Π_g is 1-formal (see §11.1 for more on this). In particular, there is a Taylor expansion $\Pi_g \rightarrow \widehat{\mathrm{gr}}(\mathbb{k}G)$, for any field \mathbb{k} of characteristic 0. In [50], Lin constructed explicitly such an expansion for $\mathbb{k} = \mathbb{C}$, using Chen’s iterated integrals. More recently, Massuyeau [58] constructed a rational Taylor expansion of $\Pi_g = F_{2g}/\langle r \rangle$, by suitably deforming the power series expansion of the free group F_{2g} .

9. DERIVED SERIES AND LIE ALGEBRAS

We now study some of the relationships between the derived series of a group and the derived series of the corresponding Lie algebras.

9.1. Derived series. Consider the derived series of a group G , starting at $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Note that any homomorphism $\phi: G \rightarrow H$ takes $G^{(i)}$ to $H^{(i)}$. The quotient groups, $G/G^{(i)}$, are solvable; in particular, $G/G' = G_{\mathrm{ab}}$, while G/G'' is the maximal metabelian quotient of G .

Assume G is a finitely generated group, and fix a coefficient field \mathbb{k} of characteristic 0.

Proposition 9.1 ([80]). *The holonomy Lie algebras $\mathfrak{h}(G/G^{(i)}; \mathbb{k})$ of the derived quotients of G are isomorphic to $\mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})'$ for $i = 1$, and are isomorphic to $\mathfrak{h}(G; \mathbb{k})$ for $i \geq 2$.*

The next theorem is the Lie algebra version of Theorem 3.5 from [63].

Theorem 9.2 ([63]). *For each $i \geq 2$, there is an isomorphism of complete, separated filtered Lie algebras,*

$$\mathfrak{m}(G/G^{(i)}; \mathbb{k}) \cong \mathfrak{m}(G; \mathbb{k}) / \overline{\mathfrak{m}(G; \mathbb{k})^{(i)}},$$

where $\overline{\mathfrak{m}(G; \mathbb{k})^{(i)}}$ denotes the closure of $\mathfrak{m}(G; \mathbb{k})^{(i)}$ with respect to the filtration topology on $\mathfrak{m}(G; \mathbb{k})$.

9.2. Chen Lie algebras. As before, let G be a finitely generated group. For each $i \geq 2$, the i -th Chen Lie algebra of G is defined to be the associated graded Lie algebra of the corresponding solvable quotient, $\text{gr}(G/G^{(i)}; \mathbb{k})$. Clearly, this construction is functorial.

The quotient map, $q_i: G \twoheadrightarrow G/G^{(i)}$, induces a surjective morphism $\text{gr}(q_i)$ between associated graded Lie algebras $\text{gr}_k(G; \mathbb{k})$ and $\text{gr}_k(G/G^{(i)}; \mathbb{k})$. Plainly, this morphism is the canonical identification in degree 1. In fact, the map $\text{gr}(q_i)$ is an isomorphism for each $k \leq 2^i - 1$, see [80].

We now specialize to the case when $i = 2$, which is the case originally studied by K.-T. Chen in [14]. The Chen ranks of G are defined as $\theta_k(G) := \dim_{\mathbb{k}}(\text{gr}_k(G/G''; \mathbb{k}))$. For a free group F_n of rank n , Chen showed that

$$(53) \quad \theta_k(F_n) = (k-1) \binom{n+k-2}{k},$$

for all $k \geq 2$. Let us also define the *holonomy Chen ranks* of G as $\bar{\theta}_k(G) = \dim_{\mathbb{k}}(\mathfrak{h}/\mathfrak{h}'')_k$, where $\mathfrak{h} = \mathfrak{h}(G; \mathbb{k})$. It is readily seen that $\bar{\theta}_k(G) \geq \theta_k(G)$, with equality for $k \leq 2$.

9.3. Chen Lie algebras and formality. We are now ready to state and prove the main result of this section, which (together with the first corollary following it) proves Theorem 1.7 from the Introduction.

Theorem 9.3. *Let G be a finitely generated group. For each $i \geq 2$, the quotient map $q_i: G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$\Psi_G^{(i)}: \text{gr}(G; \mathbb{k}) / \text{gr}(G; \mathbb{k})^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k}).$$

Moreover, if G is a filtered-formal group, then $\Psi_G^{(i)}$ is an isomorphism and the solvable quotient $G/G^{(i)}$ is filtered-formal.

Proof. The map $q_i: G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism $\text{gr}(q_i)$ of graded \mathbb{k} -Lie algebras $\text{gr}(G; \mathbb{k})$ and $\text{gr}(G/G^{(i)}; \mathbb{k})$. By Proposition 5.1, this epimorphism factors through an isomorphism, $\text{gr}(G; \mathbb{k}) / \widetilde{\text{gr}}(G^{(i)}; \mathbb{k}) \xrightarrow{\cong} \text{gr}(G/G^{(i)}; \mathbb{k})$, where $\widetilde{\text{gr}}$ denotes the graded Lie algebra associated with the filtration $\widetilde{\Gamma}_k G^{(i)} = \Gamma_k G \cap G^{(i)}$.

On the other hand, as shown by Labute in [42, Prop. 10], the Lie ideal $\text{gr}(G; \mathbb{k})^{(i)}$ is contained in $\widetilde{\text{gr}}(G^{(i)}; \mathbb{k})$. Therefore, the map $\text{gr}(q_i)$ factors through the claimed epimorphism $\Psi_G^{(i)}$, as indicated in

the following commuting diagram,

$$(54) \quad \begin{array}{ccc} \mathrm{gr}(G; \mathbb{k}) & & \\ \downarrow & \searrow \mathrm{gr}(q_i) & \\ \mathrm{gr}(G; \mathbb{k}) / \mathrm{gr}(G; \mathbb{k})^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \mathrm{gr}(G/G^{(i)}; \mathbb{k}) \\ \downarrow & \nearrow \simeq & \\ \mathrm{gr}(G; \mathbb{k}) / \widehat{\mathrm{gr}}(G^{(i)}; \mathbb{k}) & & \end{array}$$

Suppose now that G is filtered-formal, and set $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ and $\mathfrak{g} = \mathrm{gr}(G; \mathbb{k})$. We may identify $\widehat{\mathfrak{g}} \cong \mathfrak{m}$. Let $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ be the inclusion into the completion. Passing to solvable quotients, we obtain a morphism of filtered Lie algebras,

$$(55) \quad \varphi^{(i)}: \mathfrak{g}/\mathfrak{g}^{(i)} \longrightarrow \mathfrak{m}/\overline{\mathfrak{m}^{(i)}}.$$

Passing to the associated graded Lie algebras, we obtain the following diagram:

$$(56) \quad \begin{array}{ccc} \mathfrak{g}/\mathfrak{g}^{(i)} & \xrightarrow{\Psi_G^{(i)}} & \mathrm{gr}(G/G^{(i)}; \mathbb{k}) \\ \downarrow \mathrm{gr}(\varphi^{(i)}) & & \downarrow \cong \\ \mathrm{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) & \xrightarrow{\cong} & \mathrm{gr}(\mathfrak{m}(G/G^{(i)}; \mathbb{k})). \end{array}$$

All the graded Lie algebras in this diagram are generated in degree 1, and all the morphisms induce the identity in this degree. Therefore, the diagram is commutative. Moreover, the right vertical arrow from (55) is an isomorphism by Quillen's isomorphism (42), while the lower horizontal arrow is an isomorphism by Theorem 9.2.

Recall that, by assumption, $\mathfrak{m} = \widehat{\mathfrak{g}}$; therefore, the inclusion of filtered Lie algebras $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$ induces a morphism between the following two exact sequences,

$$(57) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) & \longrightarrow & \mathrm{gr}(\mathfrak{m}) & \longrightarrow & \mathrm{gr}(\mathfrak{m})/\widehat{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) \longrightarrow 0 \\ & & \uparrow \text{dotted} & & \uparrow \cong & & \uparrow \text{dotted} \\ 0 & \longrightarrow & \mathfrak{g}^{(i)} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}^{(i)} \longrightarrow 0. \end{array}$$

Here $\widehat{\mathrm{gr}}$ means taking the associated graded Lie algebra corresponding to the induced filtration. Using formulas (2) and (7), it can be shown that $\widehat{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}}) = \mathfrak{g}^{(i)}$. Therefore, the morphism $\mathfrak{g}/\mathfrak{g}^{(i)} \rightarrow \mathrm{gr}(\mathfrak{m})/\widehat{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}})$ is an isomorphism. We also know that $\mathrm{gr}(\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}) = \mathrm{gr}(\mathfrak{m})/\widehat{\mathrm{gr}}(\overline{\mathfrak{m}^{(i)}})$. Hence, the map $\mathrm{gr}(\varphi^{(i)})$ is an isomorphism, and so, by (56), the map $\Psi_G^{(i)}$ is an isomorphism, too.

By Lemma 2.2, the map $\varphi^{(i)}$ induces an isomorphism of complete, filtered Lie algebras between the degree completion of $\mathfrak{g}/\mathfrak{g}^{(i)}$ and $\mathfrak{m}/\overline{\mathfrak{m}^{(i)}}$. As shown above, $\Psi_G^{(i)}$ is an isomorphism; hence, its degree completion is also an isomorphism. Composing with the isomorphism from Theorem 9.2, we obtain an isomorphism between the degree completion $\widehat{\mathrm{gr}}(G/G^{(i)}; \mathbb{k})$ and the Malcev Lie algebra $\mathfrak{m}(G/G^{(i)}; \mathbb{k})$. This shows that the solvable quotient $G/G^{(i)}$ is filtered-formal. \square

Remark 9.4. As shown in [42, §3], the analogue of Theorem 9.3 does not hold if the ground field \mathbb{k} has characteristic $p > 0$. More precisely, there are pro- p groups G for which the morphisms $\Psi_G^{(i)}$ ($i \geq 2$) are not isomorphisms.

Returning now to the setup from Lemma 5.6, let us compose the canonical projection $\text{gr}(q_i): \text{gr}(G; \mathbb{k}) \rightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$ with the epimorphism $\Phi_G: \mathfrak{h}(G; \mathbb{k}) \rightarrow \text{gr}(G; \mathbb{k})$. We obtain in this fashion an epimorphism $\mathfrak{h}(G; \mathbb{k}) \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$, which fits into the following commuting diagram (we will suppress the coefficient field \mathbb{k}):

$$(58) \quad \begin{array}{ccc} \mathfrak{h}(G) & \xrightarrow{\Phi_G} & \text{gr}(G) \\ \swarrow & \searrow & \downarrow \\ \mathfrak{h}(G/G^{(i)}) & \twoheadrightarrow & \text{gr}(G/G^{(i)}) \\ \searrow & \swarrow & \downarrow \\ \mathfrak{h}(G)/\mathfrak{h}(G)^{(i)} & \twoheadrightarrow & \text{gr}(G)/\text{gr}(G)^{(i)}. \end{array}$$

Putting things together, we obtain the following corollary, which recasts Theorem 4.2 from [63] in a setting which is both functorial, and holds in wider generality. This corollary provides a way to detect non-1-formality of groups.

Corollary 9.5. *For each for $i \geq 2$, there is a natural epimorphism of graded \mathbb{k} -Lie algebras,*

$$\Phi_G^{(i)}: \mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})^{(i)} \twoheadrightarrow \text{gr}(G/G^{(i)}; \mathbb{k}).$$

Moreover, if G is 1-formal, then $\Phi_G^{(i)}$ is an isomorphism.

Corollary 9.6. *Suppose the group G is 1-formal. Then, for each for $i \geq 2$, the solvable quotient $G/G^{(i)}$ is graded-formal if and only if $\mathfrak{h}(G; \mathbb{k})^{(i)}$ vanishes.*

Proof. By Proposition 9.1, the canonical projection $q_i: G \rightarrow G/G^{(i)}$ induces an isomorphism $\mathfrak{h}(q_i): \mathfrak{h}(G; \mathbb{k}) \rightarrow \mathfrak{h}(G/G^{(i)}; \mathbb{k})$. Since we assume G is 1-formal, Corollary 9.5 guarantees that the map $\Phi_G^{(i)}: \mathfrak{h}(G; \mathbb{k})/\mathfrak{h}(G; \mathbb{k})^{(i)} \rightarrow \text{gr}(G/G^{(i)}; \mathbb{k})$ is an isomorphism. The claim follows from the left square of diagram (58). \square

10. TORSION-FREE NILPOTENT GROUPS

In this section we study the graded and filtered formality properties of a well-known class of groups: that of finitely generated, torsion-free nilpotent groups. In the process, we prove Theorem 1.8 from the Introduction.

10.1. Nilpotent groups and Lie algebras. We start by reviewing the construction of the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G (see Cenkli and Porter [13], Lambe and Priddy [44], and Malcev [55] for more details). There is a refinement of the upper central series of such a group,

$$(59) \quad G = G_1 > G_2 > \cdots > G_n > G_{n+1} = 1,$$

with each subgroup $G_i < G$ a normal subgroup of G_{i+1} , and each quotient G_i/G_{i+1} an infinite cyclic group. (The integer n is an invariant of the group, called the length of G .) Using this fact, we

can choose a *Malcev basis* $\{u_1, \dots, u_n\}$ for G , which satisfies $G_i = \langle G_{i+1}, u_i \rangle$. Consequently, each element $u \in G$ can be written uniquely as $u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n}$.

Using this basis, the group G , as a set, can be identified with \mathbb{Z}^n via the map sending $u_1^{a_1} \cdots u_n^{a_n}$ to $a = (a_1, \dots, a_n)$. The multiplication in G then takes the form

$$(60) \quad u_1^{a_1} \cdots u_n^{a_n} \cdot u_1^{b_1} \cdots u_n^{b_n} = u_1^{\rho_1(a,b)} \cdots u_n^{\rho_n(a,b)},$$

where $\rho_i: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ is a rational polynomial function, for each $1 \leq i \leq n$. This procedure identifies the group G with the group (\mathbb{Z}^n, ρ) , with multiplication the map $\rho = (\rho_1, \dots, \rho_n): \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$. Thus, we can define a simply-connected nilpotent Lie group $G \otimes \mathbb{k} = (\mathbb{k}^n, \rho)$ by extending the domain of ρ , which is called the *Malcev completion* of G .

The discrete group G is a subgroup of the real Lie group $G \otimes \mathbb{R}$. The quotient space, $M = (G \otimes \mathbb{R})/G$, is a compact manifold, called a *nilmanifold*. As shown in [55], the Lie algebra of M is isomorphic to $\mathfrak{m}(G; \mathbb{R})$. It is readily apparent that the nilmanifold M is an Eilenberg–MacLane space of type $K(G, 1)$. As shown by Nomizu, the cohomology ring $H^*(M, \mathbb{R})$ is isomorphic to the cohomology ring of the Lie algebra $\mathfrak{m}(G; \mathbb{R})$.

The polynomial functions ρ_i have the form

$$(61) \quad \rho_i(a, b) = a_i + b_i + \tau_i(a_1, \dots, a_{i-1}, b_1, \dots, b_{i-1}).$$

Denote by $\sigma = (\sigma_1, \dots, \sigma_n)$ the quadratic part of ρ . Then \mathbb{k}^n can be given a Lie algebra structure, with bracket $[a, b] = \sigma(a, b) - \sigma(b, a)$. As shown in [44], this Lie algebra is isomorphic to the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k})$.

The group (\mathbb{Z}^n, ρ) has canonical basis $\{e_i\}_{i=1}^n$, where e_i is the i -th standard basis vector. Then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = (\mathbb{k}^n, [\ , \])$ has Lie bracket given by $[e_i, e_j] = \sum_{k=1}^n s_{i,j}^k e_k$, where $s_{i,j}^k = b_k(e_i, e_j) - b_k(e_j, e_i)$.

The Chevalley–Eilenberg complex $\wedge^*(\mathfrak{m}(G; \mathbb{k}))$ is a minimal model for $M = K(G, 1)$. Clearly, this model is generated in degree 1; thus, it is also a 1-minimal model for G . As shown by Hasegawa in [36], the nilmanifold M is formal if and only if M is a torus.

10.2. Nilpotent groups and filtered formality. Let G be a finitely generated, torsion-free nilpotent group, and let $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ be its Malcev Lie algebra, as described above. Note that $\text{gr}(\mathfrak{m}) = \mathbb{k}^n$ has the same basis e_1, \dots, e_n as \mathfrak{m} , but, as we shall see, the Lie bracket on $\text{gr}(\mathfrak{m})$ may be different. The Lie algebra \mathfrak{m} (and thus, the group G) is filtered-formal if and only if $\mathfrak{m} \cong \widehat{\text{gr}}(\mathfrak{m}) = \text{gr}(\mathfrak{m})$, as filtered Lie algebras. In general, though, this isomorphism need not preserve the chosen basis.

Example 10.1. For any finitely generated free group F , the k -step, free nilpotent group $F/\Gamma_{k+1}F$ is filtered-formal. Indeed, F is 1-formal, and thus filtered-formal. Hence, by Theorem 7.14, each nilpotent quotient of F is also filtered-formal. In fact, as shown in [58, Cor. 2.14], $\mathfrak{m}(F/\Gamma_{k+1}F) \cong \mathbf{L}/(\Gamma_{k+1}\mathbf{L})$, where $\mathbf{L} = \text{lie}(F)$.

Example 10.2. Let G be the 3-step, rank 2 free nilpotent group $F_2/\Gamma_4 F_2$. Identifying G with \mathbb{Z}^5 as a set, then the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = \mathbb{k}^5$ has Lie brackets given by $[e_1, e_2] = e_3 - e_4/2 - e_5$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, and $[e_i, e_j] = 0$, otherwise (see [44, 13]). It is readily checked that the identity map of \mathbb{k}^5 is not a Lie algebra isomorphism between $\mathfrak{m} = \mathfrak{m}(G; \mathbb{k})$ and $\text{gr}(\mathfrak{m})$. Moreover, the differential of the 1-minimal model $\mathcal{M}(G) = \wedge^*(\mathfrak{m})$ is not homogeneous on the Hirsch weights, although \mathfrak{m} (and G) are filtered-formal.

Now consider a finite-dimensional, nilpotent Lie algebra \mathfrak{m} over a field \mathbb{k} of characteristic 0. It is readily seen that the filtered-formality of such a Lie algebra coincides with the notions of ‘Carnot’, ‘naturally graded’, ‘homogeneous’ and ‘quasi-cyclic’ which appear in [16, 17, 39, 40, 48].

The question whether the Carnot property descends from $\mathbb{k} = \mathbb{R}$ to \mathbb{Q} was first raised by Johnson in [39]. A positive answer was given by Dekimpe and Lee in [17, Cor. 4.2], but, as Cornulier points out in [16, Rem. 3.15] their proof has a gap. Nevertheless, in [16, Thm. 3.14], Cornulier gives a complete solution to Johnson’s question. Theorem 2.8 allows us to recover Cornulier’s result by completely different means.

Corollary 10.3 ([16]). *Let \mathfrak{m} be a finite-dimensional, nilpotent Lie algebra over a field \mathbb{k} of characteristic 0, and let $\mathbb{k} \subset \mathbb{K}$ be a field extension. Then \mathfrak{m} is Carnot over \mathbb{k} if and only if $\mathfrak{m} \otimes_{\mathbb{k}} \mathbb{K}$ is Carnot over \mathbb{K} .*

10.3. Torsion-free nilpotent groups and filtered formality. We now study in more detail the filtered-formality properties of torsion-free nilpotent groups. We start by singling out a rather large class of groups which enjoy this property.

Theorem 10.4. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then G is filtered-formal.*

Proof. The lower central series of our group takes the form $G = \Gamma_1 G > \Gamma_2 G > \Gamma_3 G = 1$. Let $\{x_1, \dots, x_n\}$ be a basis for $G/\Gamma_2 G = \mathbb{Z}^n$, and let $\{y_1, \dots, y_m\}$ be a basis for $\Gamma_2 G = \mathbb{Z}^m$. Then, as shown for instance by Igusa and Orr in [38, Lemma 6.1], the group G has presentation

$$(62) \quad G = \left\langle x_1, \dots, x_n, y_1, \dots, y_m \mid [x_i, x_j] = \prod_{k=1}^m y_k^{c_{i,j}^k}, [y_i, y_j] = 1, \text{ for } i < j; [x_i, y_j] = 1 \right\rangle.$$

Let $a, b \in \mathbb{Z}^{n+m}$. A routine computation shows that

$$(63) \quad \rho_i(a, b) = a_i + b_i, \quad \text{for } 1 \leq i \leq n,$$

$$\rho_{n+k}(a, b) = a_{n+k} + b_{n+k} - \sum_{j=1}^k \sum_{i=j+1}^n c_{j,i}^k a_i b_j, \quad \text{for } 1 \leq k \leq m.$$

Set $c_{j,i}^k = -c_{i,j}^k$ if $j > i$. It follows that the Malcev Lie algebra $\mathfrak{m}(G; \mathbb{k}) = (\mathbb{k}^{n+m}, [\cdot, \cdot])$ has Lie bracket given on generators by $[e_i, e_j] = \sum_{k=1}^m c_{i,j}^k e_{n+k}$ for $1 \leq i \neq j \leq n$, and zero otherwise.

Turning now to the associated graded Lie algebra of our group, we have an additive decomposition, $\text{gr}(G; \mathbb{k}) = \text{gr}_1(G; \mathbb{k}) \oplus \text{gr}_2(G; \mathbb{k}) = \mathbb{k}^n \oplus \mathbb{k}^m$, where the first factor has basis $\{e_1, \dots, e_n\}$, the second factor has basis $\{e_{n+1}, \dots, e_{n+m}\}$, and the Lie bracket is given as above. Therefore, $\mathfrak{m}(G; \mathbb{k}) \cong \text{gr}(G; \mathbb{k})$, as filtered Lie algebras. Hence, G is filtered-formal. \square

It is known that all nilpotent Lie algebras of dimension 4 or less are filtered-formal, see for instance [16]. In general, though, finitely generated, torsion-free nilpotent groups need not be filtered-formal. We illustrate this phenomenon with two examples: the first one extracted from the work of Cornulier [16], and the second one adapted from the work of Lambe and Priddy [44]. In both examples, the nilpotent Lie algebra \mathfrak{m} in question may be realized as the Malcev Lie algebra of a finitely generated, torsion-free nilpotent group G .

Example 10.5. Let \mathfrak{m} be the 5-dimensional \mathbb{k} -Lie algebra with non-zero Lie brackets given by $[e_1, e_3] = e_4$ and $[e_1, e_4] = [e_2, e_3] = e_5$. It is readily checked that the center of \mathfrak{m} is 1-dimensional,

generated by e_5 , while the center of $\text{gr}(\mathfrak{m})$ is 2-dimensional, generated by e_2 and e_5 . Therefore, $\mathfrak{m} \not\cong \text{gr}(\mathfrak{m})$, and so \mathfrak{m} is not filtered-formal. It follows that the nilpotent group G is not filtered-formal, either. From Theorem 6.5, it readily checked that the 1-minimal model $\mathcal{M}(G) = \bigwedge^*(\mathfrak{m})$ does not have positive Hirsch weights, nevertheless, $\mathcal{M}(G)$ has positive weights, given by the index of the chosen basis.

Example 10.6. Let \mathfrak{m} be the 7-dimensional \mathbb{k} -Lie algebra with non-zero Lie brackets given on basis elements by $[e_2, e_3] = e_6$, $[e_2, e_4] = e_7$, $[e_2, e_5] = -e_7$, $[e_3, e_4] = e_7$, and $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Then $\text{gr}(\mathfrak{m})$ has the same additive basis as \mathfrak{m} , with non-zero brackets given by $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq 6$. Plainly, $\text{gr}(\mathfrak{m})$ is metabelian, (i.e., its derived subalgebra is abelian), while \mathfrak{m} is not metabelian. Thus, once again, $\mathfrak{m} \not\cong \text{gr}(\mathfrak{m})$, and so both \mathfrak{m} and G are not filtered-formal. In this case, though, we cannot use the indexing of the basis to put positive weights on $\mathcal{M}(G)$.

In view of Theorem 8.5, the nilpotent groups from Examples 10.5 and 10.6 do not admit any Taylor expansions.

10.4. Filtered formality and Koszulness. Carlson and Toledo [11] classified finitely generated, 1-formal, nilpotent groups with first Betti number 5 or less, while Plantiko [69] gave sufficient conditions for the associated graded Lie algebras of such groups to be non-quadratic. The following proposition follows from Theorem 4.1 in [69] and Lemma 2.4 in [11].

Proposition 10.7 ([11, 69]). *Let $G = F/R$ be a finitely presented, torsion-free, nilpotent group. If there exists a non-zero decomposable element in the kernel of the cup product map $H^1(G; \mathbb{k}) \wedge H^1(G; \mathbb{k}) \rightarrow H^2(G; \mathbb{k})$, then G is not graded-formal.*

Here an element $u \in H^2(G; \mathbb{k})$ is said to decomposable if $u = v \wedge w$ for some $v, w \in H^1(G; \mathbb{k})$.

Example 10.8. Let $U_n(\mathbb{R})$ be the nilpotent Lie group of upper triangular matrices with 1's along the diagonal. The quotient $M = U_n(\mathbb{R})/U_n(\mathbb{Z})$ is a nilmanifold of dimension $N = n(n-1)/2$. The unipotent group $U_n(\mathbb{Z})$ has canonical basis $\{u_{ij} \mid 1 \leq i < j \leq n\}$, where u_{ij} is the matrix obtained from the identity matrix by putting 1 in position (i, j) . Moreover, $U_n(\mathbb{Z}) \cong (\mathbb{Z}^N, \rho)$, where $\rho_{ij}(a, b) = a_{ij} + b_{ij} + \sum_{i < k < j} a_{ik} b_{kj}$, see [44]. The unipotent group $U_n(\mathbb{Z})$ is filtered-formal; nevertheless, Proposition 10.7 shows that this group is not graded-formal for $n \geq 3$.

Proposition 10.9. *Let G be a finitely generated, torsion-free, nilpotent group, and suppose G is filtered-formal. Then G is abelian if and only if the algebra $U(\text{gr}(G; \mathbb{k}))$ is Koszul.*

Proof. We only need to prove the non-trivial direction. If the algebra $U = U(\text{gr}(G; \mathbb{k}))$ is Koszul, then the Lie algebra $\text{gr}(G; \mathbb{k})$ is quadratic, i.e., the group G is graded-formal. Under the assumption that G is filtered-formal, we then have that G is 1-formal.

Let M be the nilmanifold with fundamental group G . Then M is also 1-formal. By Nomizu's theorem, the cohomology ring $A = H^*(M; \mathbb{k})$ is isomorphic to the Yoneda algebra $\text{Ext}_U^*(\mathbb{k}, \mathbb{k})$. On the other hand, since U is Koszul, the Yoneda algebra is isomorphic to $U^!$, which is also Koszul. Hence, A is a Koszul algebra. As shown by Papadima and Yuzvinsky [67], if M is 1-formal and if A is Koszul, then M is formal. By [36], this happens if and only if M is a torus. This completes the proof. \square

Corollary 10.10. *Let G be a finitely generated, torsion-free, 2-step nilpotent group. If G_{ab} is torsion-free, then $U(\text{gr}(G; \mathbb{k}))$ is not Koszul.*

Example 10.11. Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_1, x_2][x_3, x_4] \rangle$. The group G is a 2-step, commutator-relators nilpotent group. Hence, by the above corollary, the enveloping algebra $U(\mathfrak{h}(G; \mathbb{k}))$ is not Koszul. In fact, $U(\mathfrak{h}(G; \mathbb{k}))^1$ is isomorphic to the quadratic algebra from Example 3.9, which is not Koszul.

11. SEIFERT FIBERED MANIFOLDS

We now use our techniques to study the fundamental groups of orientable Seifert manifolds from a rational homotopy viewpoint. We start our analysis with the fundamental groups of Riemann surfaces.

11.1. Riemann surfaces. Let Σ_g be the closed, orientable surface of genus g . The fundamental group $\Pi_g = \pi_1(\Sigma_g)$ is a 1-relator group, with generators $x_1, y_1, \dots, x_g, y_g$ and a single relation, $[x_1, y_1] \cdots [x_g, y_g] = 1$. Since this group is trivial for $g = 0$, we will assume for the rest of this subsection that $g > 0$.

The cohomology algebra $A = H^*(\Sigma_g; \mathbb{k})$ is the quotient of the exterior algebra on generators $a_1, b_1, \dots, a_g, b_g$, in degree 1 by the ideal I generated by $a_i b_i - a_j b_j$, for $1 \leq i < j \leq g$, together with $a_i a_j, b_i b_j, a_i b_j, a_j b_i$, for $1 \leq i < j \leq g$. It is readily seen that the generators of I form a quadratic Gröbner basis for this ideal; therefore, A is a Koszul algebra.

The Riemann surface Σ_g is a compact Kähler manifold, and thus, a formal space. It follows from Theorem 4.20 that the minimal model of Σ_g is generated in degree one, i.e., $\mathcal{M}(\Sigma_g) = \mathcal{M}(\Sigma_g, 1)$. The formality of Σ_g also implies the 1-formality of Π_g . As a consequence, the associated graded Lie algebra $\text{gr}(\Pi_g; \mathbb{k})$ is isomorphic to the holonomy Lie algebra $\mathfrak{h}(\Pi_g; \mathbb{k})$. Using again the fact that A is a Koszul algebra, we deduce from Corollary 5.5 that $\prod_{k \geq 1} (1 - t^k)^{\phi_k(\Pi_g)} = 1 - 2gt + t^2$.

11.2. Seifert fibered spaces. We will consider here only orientable, closed Seifert manifolds with orientable base. Every such manifold M admits an effective circle action, with orbit space an orientable surface of genus g , and finitely many exceptional orbits, encoded in pairs of coprime integers $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$ with $\alpha_j \geq 2$. The obstruction to trivializing the bundle $\eta: M \rightarrow \Sigma_g$ outside tubular neighborhoods of the exceptional orbits is given by an integer $b = b(\eta)$. A standard presentation for the fundamental group of M in terms of the Seifert invariants is given by

$$(64) \quad \pi_\eta := \pi_1(M) = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_s, h \mid h \text{ central}, \\ [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = h^b, z_i^{\alpha_i} h^{\beta_i} = 1 \ (i = 1, \dots, s) \rangle.$$

For instance, if $s = 0$, the corresponding manifold, $M_{g,b}$, is the S^1 -bundle over Σ_g with Euler number b . Let $\pi_{g,b} := \pi_1(M_{g,b})$ be the fundamental group of this manifold. If $b = 0$, then $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, whereas if $b = 1$, then $\pi_{g,1} = \langle x_1, y_1, \dots, x_g, y_g, h \mid [x_1, y_1] \cdots [x_g, y_g] = h, h \text{ central} \rangle$. In particular, $M_{1,1}$ is the Heisenberg 3-dimensional nilmanifold and $\pi_{1,1}$ is the group from Example 7.8.

11.3. Minimal model. As shown by Scott in [75], the Euler number $e(\eta)$ of the Seifert bundle $\eta: M \rightarrow \Sigma_g$ satisfies $e(\eta) = -b(\eta) - \sum_{i=1}^s \beta_i / \alpha_i$. If the base of the Seifert bundle has genus 0, the group π_η has first Betti number 0 or 1, according to whether $e(\eta)$ is non-zero or 0. Thus, π_η is 1-formal, and the Malcev Lie algebra $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is either 0, or the completed free Lie algebra of rank 1. To analyze the case when $g > 0$, we will employ the minimal model of M , as constructed by Putinar in [72].

Theorem 11.1 ([72]). *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. The minimal model $\mathcal{M}(M)$ is the Hirsch extension $\mathcal{M}(\Sigma_g) \otimes_{\mathbb{K}} (\wedge(c), d)$, where the differential is given by $d(c) = 0$ if $e(\eta) = 0$, and $d(c) \in \mathcal{M}^2(\Sigma_g)$ represents a generator of $H^2(\Sigma_g; \mathbb{K})$ if $e(\eta) \neq 0$.*

More precisely, recall that Σ_g is formal, and so there is a quasi-isomorphism $f: \mathcal{M}(\Sigma_g) \rightarrow (H^*(\Sigma_g; \mathbb{K}), d = 0)$. Thus, there is an element $a \in \mathcal{M}^2(M)$ such that $d(a) = 0$ and $f^*([a]) \neq 0$ in $H^2(\Sigma_g; \mathbb{K}) = \mathbb{K}$. We then set $d(c) = a$ in the second case.

To each Seifert fibration $\eta: M \rightarrow \Sigma_g$ as above, let us associate the S^1 -bundle $\bar{\eta}: M_{g, \epsilon(\eta)} \rightarrow \Sigma_g$, where $\epsilon(\eta) = 0$ if $e(\eta) = 0$, and $\epsilon(\eta) = 1$ if $e(\eta) \neq 0$. For instance, $M_{0,0} = S^2 \times S^1$ and $M_{0,1} = S^3$. The above theorem implies that

$$(65) \quad \mathcal{M}(M) \cong \mathcal{M}(M_{g, \epsilon(\eta)}).$$

Corollary 11.2. *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space. The Malcev Lie algebra of the fundamental group $\pi_\eta = \pi_1(M)$ is given by $\mathfrak{m}(\pi_\eta; \mathbb{K}) \cong \mathfrak{m}(\pi_{g, \epsilon(\eta)}; \mathbb{K})$.*

Proof. The case $g = 0$ follows from the above discussion, while the case $g > 0$ follows from (65). \square

Corollary 11.3. *Let $\eta: M \rightarrow \Sigma_g$ be an orientable Seifert fibered space with $g > 0$. Then M admits a minimal model with positive Hirsch weights.*

Proof. We know from §11.1 that the minimal model $\mathcal{M}(\Sigma_g)$ is formal, and generated in degree one (since $g > 0$). By Theorem 6.5, $\mathcal{M}(\Sigma_g)$ is isomorphic to a minimal model of Σ_g with positive Hirsch weights; denote this model by $\mathcal{H}(\Sigma_g)$.

By Theorem 11.1 and Lemma 4.3, the Hirsch extension $\mathcal{H}(\Sigma_g) \otimes_{\mathbb{K}} \wedge(c)$ is a minimal model for M , generated in degree one. Moreover, the weight of c equals 1 if $e(\eta) = 0$, and equals 2 if $e(\eta) \neq 0$. Clearly, the differential d is homogeneous with respect to these weights, and this completes the proof. \square

Corollary 11.4. *Fundamental groups of orientable Seifert manifolds are filtered-formal.*

Proof. The claim follows at once from Theorem 6.5 and Corollary 11.3. Alternatively, the claim also follows from Theorem 11.6 and the definition of filtered-formality. \square

Using Theorem 11.1 and Lemma 4.3 again, we obtain a quadratic model for the Seifert manifold M in the case when the base has positive genus.

Corollary 11.5. *Suppose $g > 0$. Then M has a quadratic model of the form $(H^*(\Sigma_g; \mathbb{K}) \otimes \wedge(c), d)$, where $\deg(c) = 1$ and the differential d is given by $d(a_i) = d(b_i) = 0$ for $1 \leq i \leq g$, $d(c) = 0$ if $e(\eta) = 0$, and $d(c) = a_1 \wedge_{\mathbb{K}} b_1$ if $e(\eta) \neq 0$.*

11.4. Malcev Lie algebra. We give now an explicit presentation for the Malcev Lie algebra of π_η as the degree completion of a certain graded Lie algebra.

Theorem 11.6. *The Malcev Lie algebra of π_η is the degree completion of the graded Lie algebra*

$$(66) \quad L(\pi_\eta) = \begin{cases} \text{lie}(x_1, y_1, \dots, x_g, y_g, z) / \langle \sum_{i=1}^g [x_i, y_i] = 0, z \text{ central} \rangle & \text{if } e(\eta) = 0; \\ \text{lie}(x_1, y_1, \dots, x_g, y_g, w) / \langle \sum_{i=1}^g [x_i, y_i] = w, w \text{ central} \rangle & \text{if } e(\eta) \neq 0, \end{cases}$$

where $\deg(w) = 2$ and the other generators have degree 1. Moreover, $\text{gr}(\pi_\eta; \mathbb{K}) \cong L(\pi_\eta)$.

Proof. The case $g = 0$ was already dealt with, so let us assume $g > 0$. There are two cases to consider.

If $e(\eta) = 0$, Corollary 11.2 says that $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is isomorphic to the Malcev Lie algebra of $\pi_{g,0} = \Pi_g \times \mathbb{Z}$, which is a 1-formal group. Furthermore, we know that $\mathrm{gr}(\Pi_g; \mathbb{k})$ is the quotient of the free Lie algebra $\mathrm{lie}(2g)$ by the ideal generated by $\sum_{i=1}^g [x_i, y_i]$. Hence, $\mathfrak{m}(\pi_\eta; \mathbb{k})$ is isomorphic to the degree completion of $\mathrm{gr}(\Pi_g \times \mathbb{Z}) = \mathrm{gr}(\Pi_g; \mathbb{k}) \times \mathrm{gr}(\mathbb{Z}; \mathbb{k})$, which is precisely the Lie algebra $L(\pi_\eta)$ from (66).

If $e(\eta) \neq 0$, Corollary 11.5 provides a quadratic model for our Seifert manifold. Taking the Lie algebra dual to this quadratic model and using [8, Thm. 4.3.6] or [4, Thm. 3.1], we obtain that the Malcev Lie algebra $\mathfrak{m}(\pi_\eta)$ is isomorphic to the degree completion of the graded Lie algebra $L(\pi_\eta)$. Furthermore, by formula (42), there is an isomorphism $\mathrm{gr}(\mathfrak{m}(\pi_\eta; \mathbb{k})) \cong \mathrm{gr}(\pi_\eta; \mathbb{k})$. This completes the proof. \square

In follow-up work [80], we give a presentation for the holonomy Lie algebra of an orientable Seifert manifold group, and derive the following result.

Proposition 11.7. *If $g = 0$, the group π_η is always 1-formal, while if $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.*

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